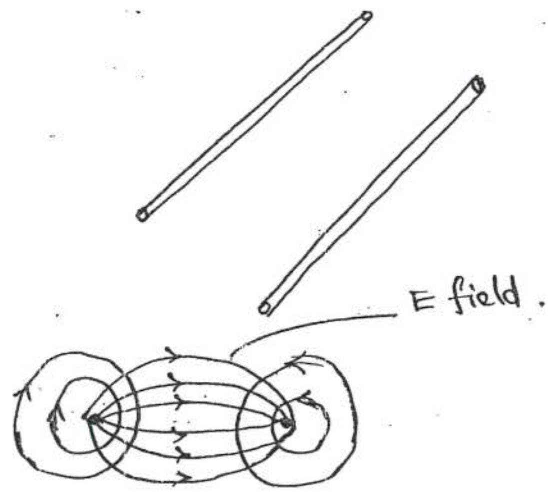
Transmission Lines

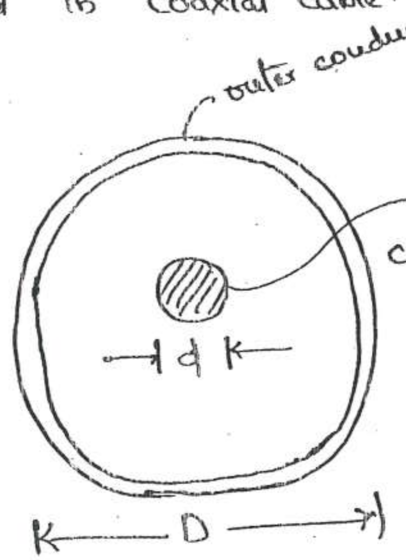
Energy can be transmitted either by radiation of free electromagnetic waves or can be varied in various conductor arrangement known as Transmission line. Transmission line is a conductive method of guiding electric energy from one place to another. They act as a link between antenna and transmitter or Receiver. They are also used as circuit elements ie like resistor, inductor or capacitor etc. Basically there are four types of Transmission lines.

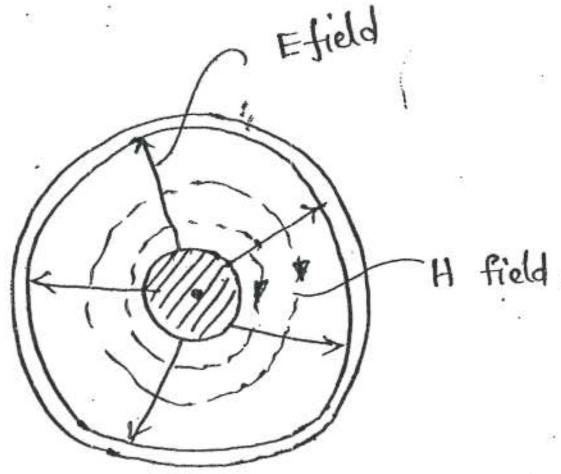
a) parallel wire type : A common form of transmission line is open wire line because of its construction. They are commonly used as telephone line telegraphy lines and power lines. short lines are used as antenna feeders and impedance Matching devices. They are very easy to construct and cheaper. The geometric structure and field lines are shown in the figure. open wire lines are not suitable for frequencies above 100 MHz.



The Electrical Energy propagating through these lines set up electric field between the conductors. These fields are at right angles to each other and to the direction of propagation. This type of transmission is known as Transverse Electromagnetic mode of propagation.

b) coaxial type :- It is entirely different in construction. In this one conductor is hollow tube and second conductor being located inside and coaxial with tube. The dielectric may be solid or gaseous. In order to avoid the severe radiation losses taken place in open wire lines at frequencies beyond 100 MHz a closed field configuration is employed in coaxial cable.





construction Details.

The advantage of coaxial cable is that electric and magnetic fields remain confined within the conductor and can not leak into freespace. Hence total radiation is eliminated. Coaxial cables are extensively used in the freq range extending upto 1 GHz and at frequencies beyond 1 GHz and these cables become unusable.

If D and d are the diameters of outer and inner conductor respectively, the primary constants of coaxial cable are given by

$$L = \frac{\mu}{2\pi} \log_e\left(\frac{D}{d}\right) \ H/m$$

$$C = 2\pi\epsilon \Big/ \log_e\left(\frac{D}{d}\right) \ F/m.$$

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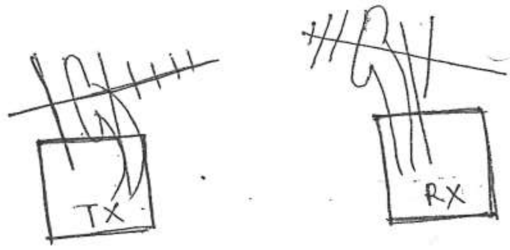


TEM Mode exists in Tx lines

Types of Transmission Lines :

1. parallel wire type

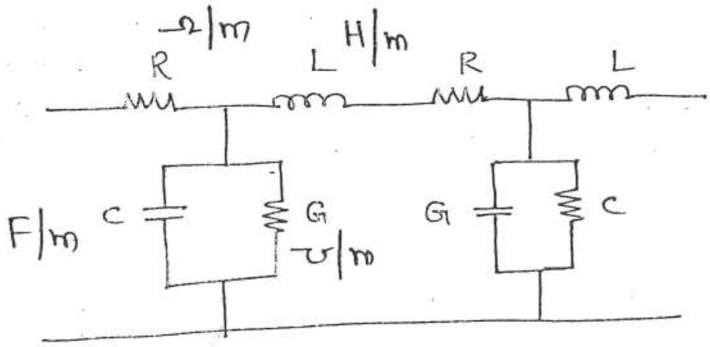
   ex : Telephone wires

2. Coaxial lines

   ex : cable wire

3. wave guides

Equivalent ckt of a Transmission line :



Here R, L, G, C are called as primary constants of a Transmission line.

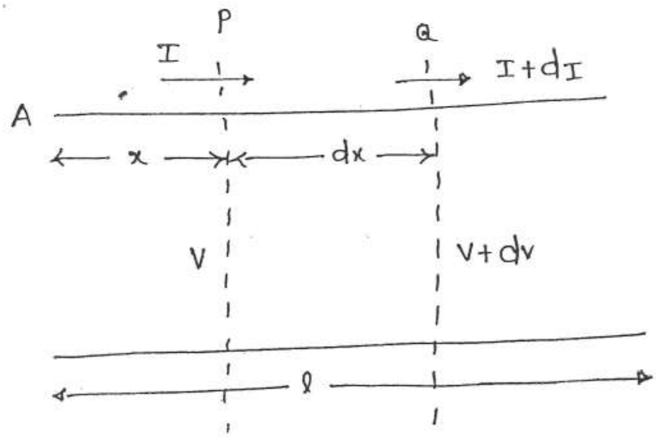
Resistance (R) : It loop resistance per unit length. units are ohms/Km.

Inductance (L) : Loop inductance per unit length. Henries/Km.

capacitance (C) : shunt capacitance between two wires. (Farad/Km)

conductance (G) : shunt conductance between two wires (mho/km).

Transmission Line Equations :-



consider a short section of line PQ of length dx at a distance x from the sending end A.

At P, Let the voltage be V and current I

At Q, let the Voltage will be V+dV and current I+dI

The Series impedance of small section dx will be $(R+J\omega L)\,dx$

The shunt admittance of " dx will be $(G+J\omega C)\,dx$

The potential difference between two points is

$$V-(V+dV) = I(R+J\omega L)\,dx$$

$$-dV = (R+J\omega L)\,I\,dx$$

$$-\frac{dV}{dx} = (R+J\omega L)\,I \qquad \text{——— (1)}$$

lly

$$I-(I+dI) = V(G+J\omega C)\,dx$$

$$\frac{-dI}{dx} = (G+J\omega C)\,V \qquad \text{——— (2)}$$

Differenting (1) and substituting (2) we have

$$\frac{d^2V}{dx^2} = (R+J\omega L)(G+J\omega C)\,V$$

$$\frac{d^2I}{dx^2} = (R+J\omega L)(G+J\omega C)\,I$$

Let $(R+j\omega L)(G+j\omega C) = P^2$

$$\therefore \frac{d^2 V}{dx^2} = P^2 V \quad \text{— (3)}$$

Here P is Propagation constant.

$$\frac{d^2 I}{dx^2} = P^2 I \quad \text{— (4)}$$

The equations 3 and 4 are called as differential equations of Transmission lines.

Solution of Transmission Line equations :- As they are 2nd order equations let the solutions will be

$$\left.\begin{aligned} V &= a e^{Px} + b e^{-Px} \quad \text{—(1)} \\ I &= c e^{px} + d e^{-px} \quad \text{—(2)} \end{aligned}\right\}$$

a, b, c, d are constants.

we know that $e^{Px} = \cosh px + \sinh px$

$$e^{-px} = \cosh px - \sinh px$$

substituting above in 1 and 2

$$\left.\begin{aligned} V &= (a+b)\cosh px + (a-b)\sinh px \\ &= A\cosh px + B\sinh px \\ \text{lly} \quad I &= C\cosh px + D\sinh px \end{aligned}\right\} \quad \begin{aligned} A &= a+b \\ B &= (a-b) \\ C &= c+d \\ D &= c-d. \end{aligned}$$

* we know that $\frac{-dv}{dx} = (R+j\omega L) I$ substituting $V = A\cosh px + B\sinh px$

$$-\frac{d}{dx}\left(A\cosh px + B\sinh px\right) = (R+j\omega L) I$$

$$-\left(AP\sinh px + BP\cosh px\right) = (R+j\omega L) I$$

Substituting $P = \sqrt{(R+j\omega L)(G+j\omega C)}$ we have

$$-\sqrt{\frac{G+j\omega C}{R+j\omega C}}\left(A\sinh px + B\cosh px\right) = I$$

Finally we have

$$\therefore \boxed{\begin{aligned} I &= \frac{-1}{Z_0}\left(A\sinh px + B\cosh px\right) \\ V &= A\cosh px + B\sinh px. \end{aligned}}$$

Determination of constants A and B :-

Let us assume that conditions at sending end are known.

Let I_s, V_s are the current and voltage at ~~receiving~~ Sending end.

At sending end $x = 0$ and $V = V_s$

we know that $V = A\cosh px + B\sinh px$

$$\therefore \quad V_s = A\cosh p(0) + B\sinh p \times 0$$

$$\boxed{V_s = A}$$

at $x = 0$ $\quad I = I_s$

$$I_s = \frac{-1}{Z_0}\left[B\cosh p \times 0 + A\sinh p \times 0\right]$$

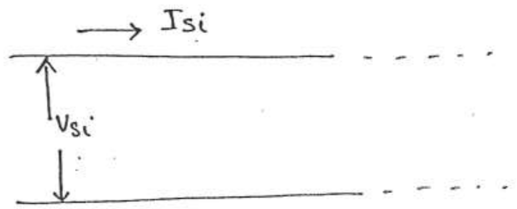
$$I_s = \frac{-1}{Z_0} B \quad \therefore \quad \boxed{B = -I_s Z_0}$$

$$\therefore \quad \boxed{V = V_s \cosh px - I_s Z_0 \sinh px}$$

$$I = \frac{-1}{Z_0}\left[-I_s Z_0 \cosh px + V_s \sinh px\right]$$

$$\boxed{I = I_s \cosh px - \frac{V_s}{Z_0}\sinh px}$$

Infinite Line :-



The input impedance (or) characteristics impedance of the line

$$Z_0 = \frac{V_{si}}{I_{si}}$$

current at any distance is $\quad I = c e^{px} + d e^{-px}$

(At sending end) at $x = 0$ $\quad I = I_{si}$ $\qquad I_{si} = c + d$

At the receiving end i.e $x = \infty$ $\quad I = 0$

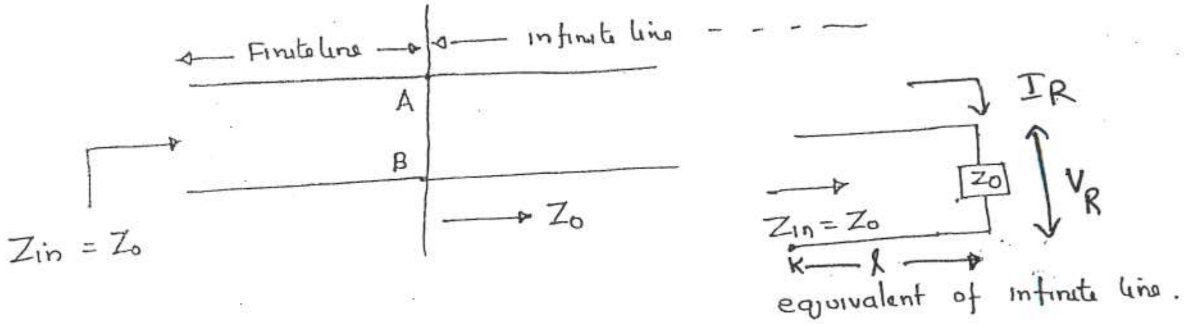
$$\therefore \quad 0 = c \cdot e^{\infty} + d e^{-\infty}$$

$$0 = c \times \infty \qquad \text{implied that} \quad c = 0$$

$$\therefore \quad \boxed{\begin{array}{l} I = I_{si}\, e^{-px} \\ V = V_{si}\, e^{-px} \end{array}}$$

similarly

Infinite Line is equivalent to a Finite Line terminated in its Z_0:



consider a length l, terminated in characteristic impedance Z_0

Let at $x = l$ $\quad V = V_R$ and $I = I_R$

$$\therefore \quad V_R = V_s \cosh pl - I_s Z_0 \sinh pl$$

$$I_R = I_s \cosh pl - \frac{V_s}{Z_0} \sinh pl$$

$$\therefore \quad Z_0 = \frac{V_R}{I_R} = \frac{V_s \cosh pl - I_s Z_0 \sinh pl}{I_s \cosh pl - \frac{V_s}{Z_0} \sinh pl}$$

Deviding both sides by Z_0

$$1 = \frac{V_s \cosh pl - I_s Z_0 \sinh pl}{Z_0 I_s \cosh pl - V_s \sinh pl}$$

$$\therefore \quad Z_0 I_s \left(\cosh pl + \sinh pl \right) = V_s \left(\cosh pl + \sinh pl \right)$$

$$Z_0 I_s = V_s$$

$$\boxed{Z_0 = \frac{V_s}{I_s} = Z_{in}}$$ Hence proved.

By definition input impedance of infinite line is Z_0.

A Finite line terminated in its Z_0 equivalent to infinite line as both have input impedance Z_0.

Secondary constants of Transmission line :-

Characterstic impedance $Z_0 = \sqrt{\dfrac{R+J\omega L}{G+J\omega C}}$

propagation const $P = \sqrt{(R+J\omega L)(G+J\omega C)}$

$= \sqrt{ZY}$

where $Z = R+J\omega L$ = series impedance

$Y = G+J\omega C$ = shunt conductance.

Characterstic impedance :

we know $-\dfrac{dv}{dx} = (R+J\omega L)\, I$

$$-\frac{d}{dx}\left(V_{si}\, e^{-px}\right) = (R+J\omega L)\, I_{si}\, e^{-px}$$

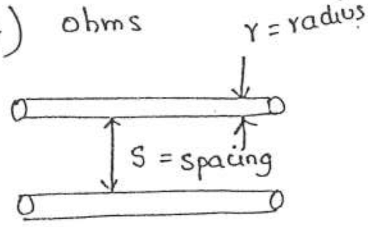
$$P.\, V_{si}\, \cancel{e^{-px}} = (R+J\omega L)\, I_{si}\, \cancel{e^{-px}}$$

$$\frac{V_{si}}{I_{si}} = \frac{R+J\omega L}{P}$$

$$Z_0 = \frac{V_{si}}{I_{si}} = \frac{R+J\omega L}{\sqrt{(R+J\omega L)(G+J\omega C)}} = \sqrt{\frac{R+J\omega L}{G+J\omega C}}$$

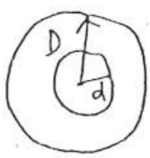
Z_0 for open wire line :

$$Z_0 = 276 \log_{10}\left(\frac{S}{r}\right) \text{ ohms}$$



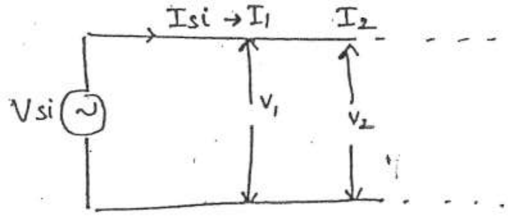
Coaxial cable :

$$Z_0 = \frac{138}{\sqrt{\epsilon_r}} \log_{10}\left(\frac{D}{d}\right) \text{ ohms}$$



Propagation constant : It governs the manner in which V and I vary with 'x'. It is a function of frequency.

definition : It is defined as the natural logarithm of the steady state vector ratio of current and voltage at any point to that at any point unit distance further from the source, when the line is infinitely long.



→ unit ← distance

$$P = \sqrt{ZY}$$

$$\therefore P = \log_e\left(\frac{I_1}{I_2}\right) = \log_e\left(\frac{V_1}{V_2}\right) \qquad P = \sqrt{(R+j\omega L)(G+j\omega C)}$$

for lossless line

R = 0 G = 0

Attenuation and phase constants :

$$P = \alpha + J\beta = \sqrt{(R+J\omega L)(G+J\omega C)} \qquad P = j\omega\sqrt{LC}$$

$$\therefore \alpha = \sqrt{\frac{1}{2}\left(RG - \omega^2 LC\right) + \sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}} = \text{Atten. const}$$

(neper/km)

$$\beta = \sqrt{\frac{1}{2}\left(\omega^2 LC - RG\right) + \sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}}$$

= phase shift const

(rad / Km)

Difinitions :

wave length : It is the distance that the wave travels along the line to have phase shift of 2π radians.

Velocity of propagation : $V_p = \dfrac{\omega}{\beta}$

Group velocity : It is the Velocity of the envelop of a complex signal. (v_g)

$$v_g = \frac{\omega_2 - \omega_1}{\beta_2 - \beta_1} = \frac{d\omega}{d\beta}$$

Condition for Minimum attenuation :-

we know that attenuation constant

$$\alpha = \sqrt{\frac{1}{2}\left(RG-\omega^2LC\right)+\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}}$$

As primary attenuation const α depends on L we have

$$\frac{d\alpha}{dL}=0 \quad \text{Let } x=\sqrt{\frac{1}{2}\left(RG-\omega^2LC\right)+\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}}$$

$$\frac{d\alpha}{dL}=\frac{1}{2\sqrt{x}}\frac{d}{dL}\left(\frac{1}{2}\left(RG-\omega^2LC\right)+\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}\right)=0$$

$$=\frac{1}{2\sqrt{x}}\left(-\omega^2C\cdot+\frac{1}{2\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}}\cdot 2\left(G^2+\omega^2C^2\right)\omega^2L\right)=0$$

$$=\frac{1}{2\sqrt{x}}\left(\frac{-\cancel{\omega^2}C\,\cancel{2}\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}+\cancel{2}\left(G^2+\omega^2C^2\right)\cancel{\omega^2}L}{2\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}}\right)=0$$

$$C\sqrt{\left(R^2+\omega^2L^2\right)\left(G^2+\omega^2C^2\right)}=L\left(G^2+\omega^2C^2\right)$$

$$\frac{L}{C}=\sqrt{\frac{R^2+\omega^2L^2}{G^2+\omega^2C^2}} \quad \text{Squaring on both sides}$$

$$L^2G^2+\cancel{\omega^2L^2C^2}=R^2C^2+\cancel{\omega^2L^2C^2}$$

$$\frac{R^2}{L^2}=G^2/C^2$$

Finally we have $\boxed{\frac{R}{L}=\frac{G}{C}}$ distortionless condition.

The characteristic impedance of a transmission line is 8039.5 Ω at a frequency of 8000 Hz. At this frequency the propagation constant was found to be $0.154\angle 90°$. Determine the values of line constant R, L, G and C.

Solution Given: $Z_0 = 8039.5\ \Omega$, $\gamma = 0.154\angle 90°$,
$f = 8$ kHz, $\omega = 2\pi f = 2\pi \times 8000 = 16000\pi$ radians/s.

We know that $R + j\omega L = Z \times \gamma = 0.154\angle 90° \times 8039.5$.

$$R + j\omega L = 1238.1\angle 90° = j1238.1.$$

$$R = 0,\ \omega L = 1238.083.$$

$$L = \frac{1238.083}{16000\pi} = 24.64 \quad \text{mH/km}.$$

Similarly, $G + j\omega C = \dfrac{\gamma}{Z_0} = \dfrac{0.154\angle 90°}{8039.5} = 1.915 \times 10^{-5}\angle 90°.$

$$G + j\omega C = j1.915 \times 10^{-5}.$$

$$G = 0,\ \omega C = 1.915 \times 10^{-5}.$$

$$C = \frac{1.915 \times 10^{-5}}{16000\pi} == 0.381 \quad \text{nF/km}.$$

Example 9.9 An open wire transmission line terminated in characteristic impedance has the following primary constants at 2 kHz: $R = 12\ \Omega$/km, $L = 4$ mH/km, $G = 1\ \mu\Omega$/km, $C = 0.005\ \mu$/km. Calculate the phase velocity and the attenuation in dB suffered by a signal in a length of 200 km.

Solution Given, $R = 12\ \Omega$/km, $L = 4$ mH/km, $G = 1\ \mu\Omega$/km,
$C = 0.005\ \mu$/km, $f = 2$ kHz. $\omega = 2\pi f = 2 \times \pi \times 2000 = 4000\,\pi$ radian/s.

Series impedance, $Z = R + j\omega L$

$$= 12 + j4000\pi \times 4 \times 10^{-3} = 12 + j50.26 = 51.67\angle 76.57°\ \Omega.$$

Similarly, shunt admittance $Y = G + j\omega C$

$$= 10^{-6} + j4000\pi \times 0.005 \times 10^{-6}.$$

$$Y = (1 + j62.83) \times 10^{-6} = 62.83 \times 10^{-6}\angle 89°\ \mho.$$

We know that $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{Z \times Y}$

$$= \sqrt{51.67\angle 76.57 \times 62.83 \times 10^{-6}\angle 89.1}$$

$$= \sqrt{322 \times 10^{-3}\angle 165.57} = 0.0567\angle 82.78$$

$$= (7.13 + j56.3) \times 10^{-3}.$$

$$\gamma = \alpha + j\beta = 0.0713 + j0.0563.$$

Attenuation constant, $\alpha = 0.00713$ nepers/km.

Phase constant, $\beta = 0.0563$ radians/km.

Velocity of propagation $v_p = \frac{\omega}{\beta} = \frac{4000\pi}{0.0563} = 2.232 \times 10^5$ km/s.

Attenuation for 200 km:

$$\alpha = 0.00713 \times 200 = 1.426 \text{ nepers} = 1.426 \times 8.686 \text{ dB} = 12.38 \text{ dB}.$$

Example 9.10 An open wire transmission line terminated in its characteristic impedance has the following primary constants at 1 kHz. $R = 6\ \Omega$/km, $L = 2$ mH/km, $G = 0.5\ \mu$ mhos/km, $C = 0.05\ \mu$ f/km. Calculate α, β and phase velocity.

Solution Given:

$R = 6\Omega$/km, $L = 2$ mH/km, $G = 0.5\,\mu$mhos/km, $C = 0.05\,\mu$f/km, $f = 1$ kHz,
$\omega = 2\pi f = 2000\pi$ radians/s.

$$Z = R + j\omega L = 6 + j12.57 = 13.93\angle 64.48^\circ.$$

$$Y = G + j\omega C$$

$$= (0.5 + j2000\pi \times 0.05) \times 10^{-6}$$

$$= 315.16 \times 10^{-6}\angle 89.9^\circ.$$

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{13.93\angle 64.48^\circ}{315.16 \times 10^{-6}\angle 89.9^\circ}} = 29.24\angle -12.71^\circ.$$

$$\gamma = \sqrt{ZY}$$

$$= \sqrt{13.93\angle 64.48^\circ \times 315.16 \times 10^{-6}\angle 154.36^\circ} = 0.06623\angle 77.2^\circ$$

$$= 0.0147 + j0.0645.$$

$$\gamma = \alpha + j\beta.$$

$\alpha = 0.0147$ nepers/km; $\beta = 0.0646$ radians/km.

Phase velocity $v_p = \frac{\omega}{\beta} = \frac{2000\pi}{0.0646} = 97263$ km/s.

Example 9.11 At 8 MHz, the characteristic impedance of a transmission line is $(40 - j2)\ \Omega$ and the propagation constant is $(0.01 + j0.18)$ per metre. Find the primary constants.

Solution Given:

Frequency $f = 8$ MHz; characteristic impedance $Z_0 = (40 - j2)\ \Omega$;
propagation constant $\gamma = (0.01 + j0.18)$ per metre.

$$\omega = 2\pi f = 16\pi \times 10^6 \text{ radians/s}.$$

We know that $Z_0 = \sqrt{\dfrac{R+j\omega L}{G+j\omega C}}$ and $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$.

$\therefore \quad Z_0\gamma = R + j\omega L.$

$$R + j\omega L = (40 - j2)(0.01 + j0.18) = 0.76 + j7.18.$$

Equating real and imaginary parts,

$$R = 0.76\ \Omega/\text{m} \quad \text{and} \quad \omega L = 7.18.$$

$$L = \frac{7.18}{\omega} = \frac{7.18}{2\pi f} = \frac{7.18}{16\pi \times 10^6}\ L = 0.143\ \mu\text{H/m}.$$

Also, $G + j\omega C = \dfrac{\gamma}{Z_0} = \dfrac{(0.01 + j0.18)}{(40 - j2)} = 4.49 \times 10^{-3} \angle 89.68^\circ$

$$= 2.49 \times 10^{-5} + j4.5 \times 10^{-3}.$$

Equating real and imaginary parts,

$$G = 2.49 \times 10^{-5}\ \mho/\text{m} \quad \text{and} \quad \omega C = 4.5 \times 10^{-3}.$$

$$C = \frac{4.5 \times 10^{-3}}{16\pi \times 10^6} = 8.95 \times 10^{-11}\ C = 89.5\ \text{pF/m}.$$

Example 9.12 A telephone wire has the following constants per loop km. Resistance is 90 Ω, capacitance, 0.062 μF, inductance, 0.001 H and leakage = 1.5×10^{-6} mhos. The line is terminated in its characteristic impedance. A frequency of 1000 Hz is applied at the sending end. Calculate: (i) the characteristic impedance, (ii) wavelength, (ii) velocity propagation.

Solution Given: $R = 90\ \Omega/\text{km}$, $C = 0.062\ \mu\text{F/km}$, $L = 0.001$ H/km, leakage conductance, $G = 1.5 \times 10^{-6}$ mhos/km, frequency, $f = 1000$ Hz. Termination impedance = Characteristic impedance

Now $\quad Z = R + j\omega L = 90 + j2\pi \times 1000 \times 0.001 = 90 + j6.283$

or $\quad Z = 90.219\angle 3.99^\circ\ \Omega.$

$$Y = G + j\omega C = 1.5 \times 10^{-6} + j2\pi \times 1000 \times 0.062 \times 10^{-6}.$$

$$Y = (1.5 + j389.557) \times 10^{-6}\ \mho = 389.56 \times 10^{-6} \angle 89.78^\circ\ \mho.$$

(i) Characteristic impedance $Z_0 = \sqrt{\dfrac{Z}{Y}}$

$$= \sqrt{\frac{90.219\angle 3.99^\circ}{389.56 \times 10^{-6} \times \angle 89.78^\circ}} = \sqrt{0.2316 \times 10^6 \angle -85.79^\circ}.$$

$$Z_0 = 481.24\angle -42.89^\circ\ \Omega.$$

(ii) Wavelength

We know that the propagation constant $\gamma = \alpha + j\beta = \sqrt{ZY}$.

$$\alpha + j\beta = \sqrt{90.219\angle 3.99^\circ \times 389.56 \times 10^{-6}\angle 89.78^\circ} = 0.1874\angle 46.89$$

$$= 0.128 + j0.1368.$$

$\therefore \quad \beta = 0.1368$ radians per km.

Then wavelength $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{0.1368} = 45.93$ km.

(iii) Velocity of propagation:

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 1000}{0.1368} = 45930 \text{ km/s}.$$

Example 9.13 A coaxial line with an outer diameter of 8 mm has 50 ohm characteristic impedance. If the dielectric constant of the insulation is 1.60. Calculate the inner diameter.

Solution Given a coaxial line of characteristic impedance $Z_0 = 50\ \Omega$, outer diameter $b = 8$ mm, $\varepsilon_r = 1.6$.

We know that for a coaxial cable, the characteristic impedance is

$$Z_0 = \frac{138}{\sqrt{\varepsilon_r}} \log_{10}(b/a),$$

where $a =$ inner diameter.

$$Z_0 = \frac{138}{\sqrt{1.6}} \log_{10}(b/a) = 109.1 \log_{10}(b/a) = 50.$$

$$\log_{10}(b/a) = 0.458 \quad \text{or} \quad b/a = 2.872.$$

$$a = \frac{8}{1.872} = 2.784 \text{ mm}.$$

The diameter of the inner conductor is 2.784 mm.

Example 9.14 The characteristic impedance of a 1 km long line is 100 ohms and is terminated in 200 ohms. It is fed with 10 V, having a source resistance of 50 ohms at $\omega = 0.3$ radian/s. Find the input voltage and current.

Solution Given: Length $l = 20$ km; characteristic impedance $Z_0 = 100$ ohms; termination impedance $Z_R = 200$ ohms; source resistance $Z_g = 50$ ohms; source voltage $V_g = 10$ V; $Z_0 = 100$ ohms.

We know that the input current is $I_s = \dfrac{V_g}{Z_0 + Z_g} = \dfrac{10}{100+50} = 66.67$ mA.

Input voltage is $V_s = Z_0 I_s = 66.67 \times 10^{-3} \times 100 = 6.67$ V.

Example 9.15 A coaxial cable has the following parameters: $Z_0 = 50$ ohms, length $= 20$ km. If the power input is 1 watt and the attenuation constant is 1.5 dB/km, find the output power of the cable provided it is terminated by Z_0. Also determine the output current.

Solution Given: Length $l = 20$ km; input power $P_s = 1$ watt; characteristic impedance $Z_0 = 50$ ohms; attenuation $\alpha = 1.5$ dB/km.

We know that if output power is P_0, then

$$\text{attenuation,} \quad \alpha = \frac{1}{l} 10\log_{10}\left(\frac{P_s}{P_0}\right) \text{ dB/km.}$$

$$1.5 = \frac{1}{20} \times 10\log_{10}\left(\frac{1}{P_0}\right).$$

$$P_0 = \frac{1}{10^3} = 1 \text{ mW.}$$

If output current is I_0, then

$$P_0 = |I_0|^2 Z_0, \quad |I_0|^2 = \frac{P_0}{Z_0}.$$

$$I_0 = \sqrt{\frac{P_0}{Z_0}} = \sqrt{\frac{1 \times 10^{-3}}{50}} = 4.47 \text{ mA.}$$

Example 9.16 A uniform transmission line operating at 1 kHz has $Z_o = 75$ ohms and propagation constant of $(0.1 + j0.2)$ m^{-1}. Find G and C.

Solution Given: Frequency $f = 1$ kHz; characteristic impedance $Z_0 = 75\ \Omega$; propagation constant $\gamma = 0.1 + j0.2$ per metre; $\omega = 2\pi f = 2000\pi$ radians/s.

We know that,

$$G + j\omega C = \frac{\gamma}{Z_0} = \frac{0.1 + j0.2}{75} = (1.33 + j2.667) \times 10^{-3}.$$

Equating real and imaginary parts,

$$G = 1.33 \times 10^{-5} \text{ ℧/m and } \quad \omega C = 2.667 \times 10^{-3}.$$

$$C = \frac{2.667 \times 10^{-3}}{2000\pi} = 0.424 \times 10^{-6} = 0.424\ \mu\text{F/m.}$$

Example 9.17 A transmission line has $R = 0.01\ \Omega$/m, $G = 1 \times 10^{-6}\ \Omega$/m, $L = 10 \times 10^{-6}$ H/m, $C = 1 \times 10^{-6}$ F/m. Find the attenuation constants, phase constants and phase velocity.

Solution Given: $R = 0.01\ \Omega$/m, $L = 10 \times 10^{-6}$ H/m; $C = 1 \times 10^{-6}$ F/m, $G = 1 \times 10^{-6}$ ℧/km.

Assume signal frequency $f = 1$ kHz. Then,

$$\omega = 2\pi f = 2000\pi \text{ radians/s}$$

$$Z = R + j\omega L = 0.01 + j2000\pi \times 10^{-6} = 0.01 + j0.0628 = 0.0636\angle 80.95°.$$

$$Y = G + j\omega C = (1 + j2000\pi) \times 10^{-6} = 6283 \times 10^{-6}\angle 89.99°.$$

We know that $\gamma = \sqrt{ZY}$

$$= \sqrt{0.0636\angle 80.95° \times 6283 \times 10^{-6}\angle 89.99°} = 0.02\angle 85.47 = 0.00158 + j0.0199.$$

$$\gamma = \alpha + j\beta.$$

$\alpha = 0.00158$ nepers/m; $\beta = 0.0199$ radians/m.

Wavelength $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{0.0199} = 315.73$ m.

Phase velocity $v_p = \lambda \times f = 315.73 \times 10^3 = 315.73$ km/s.

Example 9.19 Find the characteristic impedance, propagation constant and velocity of propagation at 10 kHz for a lossless transmission line having $L = 62\ \mu$H/m and $C = 37\ \mu$F/m.

Solution Given: $L = 62 \times 10^{-6}$ H/m, $C = 37 \times 10^{-6}$ F/m, $f = 1 \times 10^4$ Hz.

For a lossless line, $R = G = 0; Z = j\omega L; Y = j\omega C.$

Characteristic impedance for a lossless transmission line is

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{L}{C}} = \sqrt{\frac{62 \times 10^{-6}}{37 \times 10^{-6}}} = 1.3\ \Omega \text{ ohms.}$$

Propagation constant

$$\gamma = \sqrt{ZY} = j\omega\sqrt{LC} = j2\pi \times 10^4\sqrt{62 \times 10^{-6} \times 37 \times 10^{-6}}.$$

$$\gamma = \alpha + j\beta = j3.01.$$

$\alpha = 0,\ \beta = j3.01$ radians/m.

Velocity of propagation $v_p = \dfrac{1}{\sqrt{LC}} = \dfrac{\omega}{\beta} = \dfrac{2\pi \times 10^4}{3.01} = 20874$ km/s.

Example 9.20 A lossless transmission line has a capacitance of 50 μF/m and an inductance of 200 mH/m. Find the characteristic impedance for a section of a line 10 m long and 500 m long.

Solution Given: $L = 200 \times 10^{-3}$ H/m; $C = 50 \times 10^{-9}$ F/m.

For a lossless line: $R = G = 0; Z = j\omega L; Y = j\omega C.$

Characteristic impedance

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{L}{C}} = \sqrt{\frac{200 \times 10^{-3}}{50 \times 10^{-6}}} = 63.24\,\Omega.$$

Since the characteristic impedance of a lossless line is independent of line length, the characteristic impedance for a section of a line 10 m long and 500 m long is the same as

$$Z_0 = 63.24\,\Omega.$$

Example 9.21 A high frequency line has the following primary constants: $L = 1.2$ mH/km, $C = 0.05\ \mu$F/km, $R = G =$ negligible. Determine the characteristic impedance and propagation constant of the line.

Solution Given: $L = 1.2 \times 10^{-3}$ H/m, $R = G = 0$, $z = j\omega L$, $\gamma = j\omega C$, $C = 0.05 \times 10^{-6}$ F/m.

Assume that $f = 1 \times 10^6$ Hz.

Characteristic impedance

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{L}{C}} = \sqrt{\frac{12 \times 10^3}{0.05 \times 10^{-6}}} = 155\ \Omega.$$

Propagation constant

$$\gamma = \sqrt{ZY} = j\omega\sqrt{LC}$$

$$= j2\pi \times 10^6 \sqrt{1.2 \times 10^{-3} \times 0.05 \times 10^{-6}}.$$

$$\gamma = \alpha + j\beta = j48.67.$$

$$\alpha = 0; \quad \beta = j48.67 \text{ radians/km.}$$

Example 9.22 A transmission line operating at 500 MHz has $Z_0 = 80$ ohms, $\alpha = 0.04$ Np/m, $\beta = 1.5$ rad/m. Find the line parameters R, L, G and C.

Solution Given $Z_0 = 80$ ohms, $\alpha = 0.04$ nepers/m, $\beta = 1.5$ radians/m, $f = 500$ MHz.

We know that $\gamma = a + j\beta = 0.04 + j1.5$.

$$\omega = 2\pi f = 2\pi \times 500 \times 10^6 = \pi \times 10^9 \text{ radians/s.}$$

Also, $R + j\omega L = Z_0 \times \gamma = 80 \times (0.04 + j1.5) = 3.2 + j120$.

Equating real and imaginary parts,

$$R = 3.2\ \Omega/\text{m}; \quad \omega L = 120.$$

$$L = \frac{120}{\pi \times 10^9} = 38.2 \times 10^{-9} \quad \text{or} \quad L = 38.2 \text{ nH/m.}$$

Similarly, $G + j\omega L = \dfrac{\gamma}{Z_0} = \dfrac{0.04 + j1.5}{80} = 5 \times 10^{-4} + j0.01875$.

Equating real and imaginary parts,

$$G = 5 \times 10^{-4} \text{ S/m.}$$

$\omega C = 0.01875.$

$$C = \frac{0.01875}{\pi \times 10^9} = 5.968 \times 10^{-12} = 5.968 \text{ pF/m}.$$

Example 9.23 A lossy cable which has $R = 2.25\ \Omega/\text{m}$, $L = 1\ \mu\text{H/m}$, $C = 1$ pF/m and $G = 0$ operates at $f = 0.5$ GHz. Find the attenuation constant of the line.

Solution Given: $R = 2.25\ \Omega/\text{m}$, $L = 1 \times 10^{-6}$ H/m, $C = 1 \times 10^{-12}$ F/m, $G = 0$.
Signal frequency $f = 0.5 \times 10^9$ Hz; $\omega = 2\pi f = \pi \times 10^9$ radians/s.

$$Z = R + j\omega L = 2.25 + j\pi \times 10^9 \times 10^{-6} = 2.25 + j3141.6.$$

$$Y = G + j\omega C = j\pi \times 10^9 \times 10^{-12} = j3.1426 \times 10^{-3}.$$

Propagation constant $\gamma = \sqrt{ZY}$

$$= \sqrt{(2.25 + j3141.6) \times j3.1426 \times 10^{-3}} = \sqrt{9.869\angle 179.96}$$

$$= 3.1414\angle 89.98 = 0.0011 + j3.1414.$$

$$\gamma = \alpha + j\beta.$$

Attenuation constant $\alpha = 0.0011$ nepers/m.
Phase shift constant $\beta = 3.1414$ radians/m.

Example 9.24 A 75 ohm TV cable of length 20 km has attenuation 3 dB/km and is terminated with matched load. The dielectric constant of the cable is $\varepsilon = 5\varepsilon_0$ and $\mu = \mu_0$. If a 5 V source at 80 MHz is applied, find the voltage and current at the load.

Solution Given:

$$Z_0 = 75 \text{ ohms}, \ f = 80 \text{ MHz}, \ V_s = 5 \text{ V}, \ \varepsilon = 5\varepsilon_0 \text{ and } \mu = \mu_0$$

$$\alpha = 3 \text{ dB/km} = 0.3454 \text{ nepers/m}$$

$$\omega = 2\pi f = 2\pi \times 80 \times 10^6 = 16\pi \times 10^7 \text{ radians/s}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = 2\pi \times 80 \times 10^6 \sqrt{5\mu_0\varepsilon_0} = 3.746 \text{ radians/m}.$$

$$\gamma = a + j\beta = 0.3454 + j3.746.$$

We know that $V_R = V_s e^{-\gamma l}$

$$V_R = 5 \times e^{-(0.3454 + j3.746)\times 20} = 5 \times 10^{-3} e^{-j74.92} = 5 \times 10^{-3}\angle 332.6^\circ \text{ V}.$$

$$I_R = \frac{V_R}{Z_R} = \frac{5 \times 10^{-3}\angle 332.6}{75} = 66.67\angle 332.6^\circ \ \mu\text{A}.$$

Example 9.25 A transmission line has $R = 15\Omega/\text{m}$, $L = 2$ H/m, $C = 0.15 \times 10^{-6}$ F/m, $G = 1 \times 10^{-6}$ ℧/km. Find the additional inductance to give a distortionless line. Also find the propagation constant at 1 kHz.

Solution Given: $R = 15\ \Omega/\text{m}$, $L = 2$ H/m, $C = 0.15 \times 10^{-6}$ F/m, $G = 1 \times 10^{-6}$ ℧/km.

Assume that signal frequency $f = 1$ kHz, Then, $\omega = 2\pi f = 200\pi$ radians/s.

Condition for distortionless line is $L_1 G = RC$.

$$L_1 = \frac{RC}{G} = \frac{15 \times 0.15 \times 10^{-6}}{1 \times 10^{-6}} = 2.25 \text{ H/m.}$$

So, additional inductance required $L' = L_1 - L = 2.25 - 2 = 0.25$ H/m.

For distortionless line, we know that

$$\alpha = R\sqrt{\frac{C}{L_1}} = 15\sqrt{\frac{0.15 \times 10^{-6}}{2.25}} = 3.873 \times 10^{-3} \text{ nepers/m.}$$

$$\beta = \omega\sqrt{L_1 C} = 2000\pi\sqrt{2.25 \times 0.15 \times 10^{-6}} = 3.65 \text{ radians/m.}$$

Propagation constant $\gamma = \alpha + j\beta = 0.00387 + j3.65$ per m.

Example 9.26 A 50 ohm distortionless transmission line has attenuation 2.5 dB/m. The line has capacitance of 0.1 nF/m. Find R, L, G, v_p and voltage at 5 m if source voltage is 10 V.

Solution Given: $Z_0 = 50$ ohms, $f = 80$ MHz, $V_s = 10$ V, $C = 0.1$ nF/m, $\alpha = 2.5$ dB/m $= 0.2878$ nepers/m.

The condition for distortionless transmission is $RC = LG$.

$$\alpha = R\sqrt{\frac{C}{L}}, \quad Z_0 = \sqrt{\frac{L}{C}}.$$

$$\alpha = \frac{R}{Z_0}, \quad v_p = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \beta = \omega\sqrt{LC}.$$

$$L = CZ_0^2 = 0.1 \times 10^{-9} \times 50^2 = 0.25\ \mu\text{H/m.}$$

$$R = \alpha Z_0 = 0.2828 \times 50 = 14.4\ \Omega/\text{m.}$$

$$G = \frac{RC}{L} = \frac{R}{Z_0^2} = \frac{14.4}{50^2} = 5.76 \text{ m℧/m.}$$

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.25 \times 10^{-6} \times 0.1 \times 10^{-9}}} = 2 \times 10^8 \text{ m/s.}$$

We know that the voltage on the line is $V_1 = V_s e^{-\alpha l}$.

Voltage at 5 m is $V_1 = 10 \times e^{-0.2878 \times 5} = 10 \times e^{-1.439} = 2.37$ V.

<u>Line</u> with any <u>termination</u> :- Totally we have 4 types of terminations.

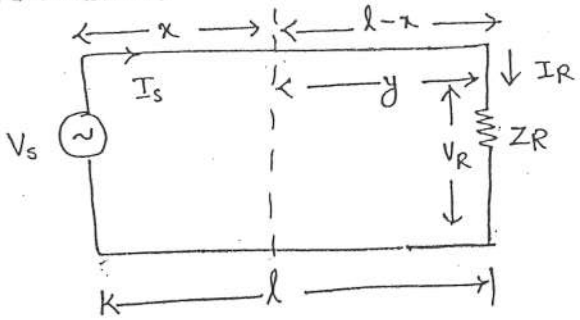
They are i) with Z_0 ii) open ckt iii) short ckt iv) Any Termination

If the line is terminated with Z_0 there is no reflection of Energy.

If the line is " $Z_0 = \infty$ (open ckt) total reflection

" $Z_0 = 0$ (short ckt) "

If the line is terminated with Z_R then there is a partial reflection and partial Transmission.



we know that $V = A\cosh px + B\sinh px$. —— (1)

$$I = \frac{-1}{Z_0}\left(A\sinh px + B\cosh px\right) \quad —(2)$$

If it is terminates with Z_R

at $x = l$ $V = V_R$ $I = I_R$

$$\therefore \quad V_R = A\cosh pl + B\sinh pl$$

$$I = \frac{-1}{Z_0}\left[A\sinh pl + B\cosh pl\right] \quad — (3)$$

$$I = \frac{-B}{Z_0}\cosh pl - \frac{A}{Z_0}\sinh pl \quad — (4)$$

To determine A multiply (1) by $\frac{\cosh pl}{Z_0}$ and 2 by $\sinh pl$ then add two equations.

$$\therefore \quad A = V_R\cosh pl + I_R Z_0 \sinh pl \quad \text{as } (\cosh^2 pl - \sinh^2 pl = 1)$$

lly $\quad B = -(V_R\sinh pl + I_R Z_0\cosh pl)$

$\therefore$ substing A and B in (1) we have

$$V = (V_R\cosh pl + I_R Z_0\sinh pl)\cosh px - (V_R\sinh pl + I_R Z_0\cosh pl)\sinh px$$

lly $\quad V = V_R\cosh p(l-x) + I_R Z_0 \sinh p(l-x)$

$$I = \frac{V_R}{Z_0}\sinh p(l-x) + I_R\cosh p(l-x)$$

as $y = l - x$

$$V = V_R \cosh py + I_R Z_0 \sinh py$$

$$I = \frac{V_R}{Z_0} \sinh py + I_R \cosh py$$

at $x = 0$ $V = V_s$ and $I = I_s$

$$\boxed{\therefore\ V_s = V_R \cosh pl + I_R Z_0 \sinh pl \qquad I_s = \frac{V_R}{Z_0} \sinh pl + I_R \cosh pl}$$

Input Impedance :- $Z_{in} = \frac{V_s}{I_s}$

$$\therefore\ Z_{in} = \frac{V_R \cosh pl + I_R Z_0 \sinh pl}{\frac{V_R}{Z_0} \sinh pl + I_R \cosh pl}$$

Multiplying Numerator and Denominator by $\frac{Z_0}{I_R}$

$$= Z_0 \cdot \left[\frac{\frac{V_R}{I_R} \cosh pl + Z_0 \sinh pl}{\frac{V_R}{I_R} \sinh pl + Z_0 \cosh pl}\right]$$

$$\boxed{Z_{in} = Z_0 \frac{Z_R \cosh pl + Z_0 \sinh pl}{Z_R \sinh pl + Z_0 \cosh pl}}$$

$$\boxed{Z_{in} = Z_0 \frac{Z_R + Z_0 \tanh pl}{Z_0 + Z_R \tanh pl}}$$

For open circuit : $Z_{in} = Z_{oc}$ and $Z_R = \infty$

$$Z_{oc} = Z_0 \frac{1 + \frac{Z_0}{Z_R} \tanh pl}{\frac{Z_0}{Z_R} + \tanh pl} = Z_0 \coth pl$$

For short circuit line : $Z_{in} = Z_{sc}$ and $Z_R = 0$

$$\therefore\ Z_{sc} = Z_0 \tanh pl$$

For Matched Termination $Z_{in} = Z_0$

For other Termination $Z_{in} \neq Z_0$.

$$\therefore\ \boxed{Z_{sc} Z_{oc} = Z_0^2}$$

$$\frac{Z_{sc}}{Z_{oc}} = \frac{Z_0 \tanh pl}{Z_0 \coth pl} = \tanh^2 pl$$

$$\tanh pl = \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

$$p = \frac{1}{l} \tanh^{-1} \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

From the above equation we can calculate Z_0, P and primary constants R, L, C, G.

Problem :- A Tx line 100 Km long gave the following results for an impedance measurement at 1796 Hz.

$$Z_{oc} = 328 \angle -29.2^\circ \text{ and } Z_{sc} = 1548 \angle 6.8^\circ .$$

Determine Line constants.

Data : $Z_{oc} = 328 \angle -29.2^\circ$ $\quad Z_{sc} = 1548 \angle 6.8^\circ$

$f = 1796$ HZ $\qquad \omega = 2\pi f$

$\therefore \quad \omega = 2 \times 3.141 \times 1796 = 11278.8$ rad/s

$$Z_0 = \sqrt{Z_{sc} Z_{oc}} = 712.56 \angle -10.2$$

$$\tanh pl = \sqrt{\frac{Z_{sc}}{Z_{oc}}} = 2.172 \angle \frac{6.8 + 29.2}{2}$$

$$\tanh pl = \frac{e^{pl} - e^{-pl}}{e^{pl} + e^{-pl}} = 2.07 + j0.6712$$

By componendo and dividendo, we get

$$\frac{e^{Pl} + e^{-Pl} + e^{Pl} - e^{-Pl}}{e^{Pl} + e^{-Pl} - e^{-Pl} + e^{-Pl}} = \frac{3.07 + j0.6712}{-1.07 - j0.6712}$$

$$e^{2Pl} = \frac{3.1425 \angle 12.33}{1.263 \angle -147.9}$$

Taking ln on both sides

$$P = \frac{1}{2l}\left\{ \ln 2.488 + j\,160.23^\circ \right\}$$

for $l = 100$ km

$$P = \frac{1}{200}\left(0.911 + j2.8\right) = 0.0147 \angle 72^\circ$$

$$R + j\omega L = \gamma \times Z_0 = 0.0147 \angle 72^\circ \times 712.56 \angle -10.2^\circ$$

$$R + j\omega L = 5.11 + j9.143$$

$$\omega l = 9.143$$

$$L = 0.81 \text{ mH/Km} \qquad R = 5.11\ \Omega/\text{Km}$$

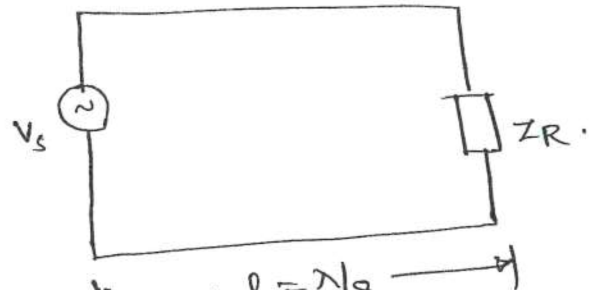
$$G + j\omega C = \frac{\gamma}{Z_0} = 2.44 \times 10^{-6} + j2048 \times 10^{-6}$$

$$G = 2.44 \times 10^{-6}\ \mho/\text{Km}$$

$$C = 0.181\ \text{nF}/\text{Km}.$$

Impedance Transformation :- Input impedance of Tx line depends on length. The important short length Tx lines are

a) Eigth wave $\left(\frac{\lambda}{8}\right)$ Tx line



we know

$$Z_{in} = Z_0 \frac{Z_R + Z_0 \tanh \gamma l}{Z_0 + Z_R \tanh \gamma l}$$

$\gamma = \alpha + j\beta \qquad \alpha = 0$

For loss less line $\alpha = 0$

$\beta = \frac{2\pi}{\lambda} \qquad \beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{8} = \frac{\pi}{4}$

$l = \lambda/8$

$$Z_{in} = Z_0 \frac{Z_R + jZ_0 \tan \beta l}{Z_0 + jZ_R \tan \beta l}$$

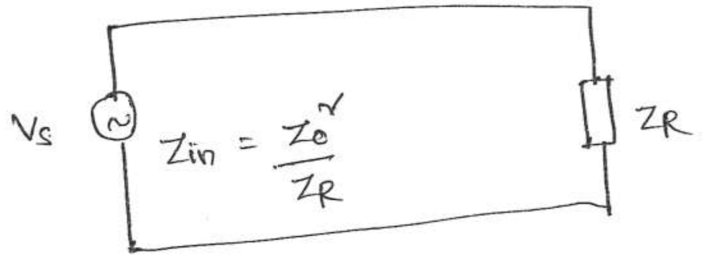
$$Z_{in} = Z_0 \left[\frac{Z_R + jZ_0}{Z_0 + jZ_R}\right]$$

$$|Z_{in}| = Z_0 \checkmark$$

$\therefore$ The Magnitude of Input impedance is equal to characteristic imped

b) Quarter wave $(\lambda/4)$ transmission line : It is nothing but impedance Transformer.

$$Z_0 = \sqrt{Z_{in} Z_R}$$



$$Z_{in} = \frac{Z_0^2}{Z_R}$$

substitute $l = \frac{\lambda}{4} \qquad \beta = 2\pi/\lambda$

$$\therefore Z_0 = \sqrt{Z_{in} Z_R}.$$

3) Half wave Tx line : $l = \lambda/2$

$$Z_{in} = Z_0 \frac{Z_R + jZ_0 \tan\beta l}{Z_0 + jZ_R \tan\beta l} \qquad l = \lambda/2 \quad \beta = 2\pi/\lambda$$

$$Z_{in} = Z_0 \frac{Z_R}{Z_0} \qquad \therefore \boxed{Z_{in} = Z_R}$$

The input Impedance of half wave line is equal to its terminating Impedance.

Insertion Loss ? It is defined as

$$IL = 20\log\left(\frac{V_1}{V_2}\right)$$

V_1 = O/p with the filter

V_2 = O/p without filter.

Transmission Lines as circuit Elements :- we Know that the input Impedance

$$Z_{in} = Z_0 \cdot \frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l}$$

For different types of terminations

$$Z_{sc} = j Z_0 \tan \beta l$$

$$Z_{oc} = j Z_0 \cot \beta l$$

Hence desired value of reactance is obtained by varying Electrical length βl of the stubs. The equivalent inductance is obtained using

$$j\omega L_{eq} = j Z_0 \tan \beta l$$

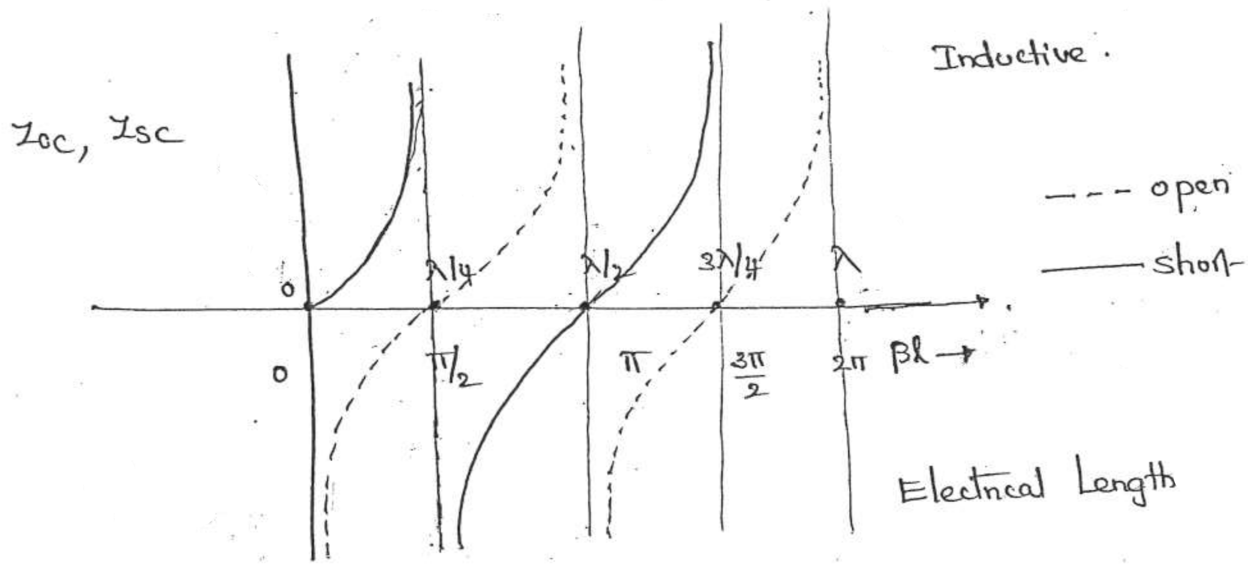
$$L_{eq} = \frac{Z_0}{\omega} \tan \beta l \quad \left(0 \le l \le \lambda/4\right)$$

Equivalent capacitance is

$$\frac{-j}{\omega C_{eq}} = j Z_0 \tan \beta l$$

$$C_{eq} = \frac{1}{\omega Z_0 \tan \beta l} \quad \left(\frac{\lambda}{4} \le l \le \lambda/2\right)$$

Variation of Z_{sc}, Z_{oc} with βl :-



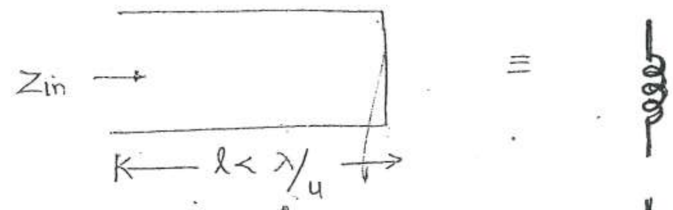
below the Horizontal line : capacitive

UHF lines as circuit Elements :-

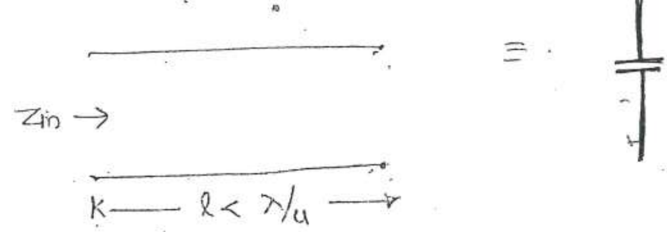
Transmission lines are not only meant for transfer of energy from one point to other point. Another application is they can be used as circuit elements at High Frequency. At above 150 MHz the ordinary lumped circuit elements become difficult to construct. At Higher frequencies the size of the transmission lines are very small. we know that the Z_{in} for Transmission line is

$$Z_{in} = Z_0 \frac{Z_R \cos\beta l + j Z_0 \sin\beta l}{Z_0 \cos\beta l + j Z_R \sin\beta l}.$$

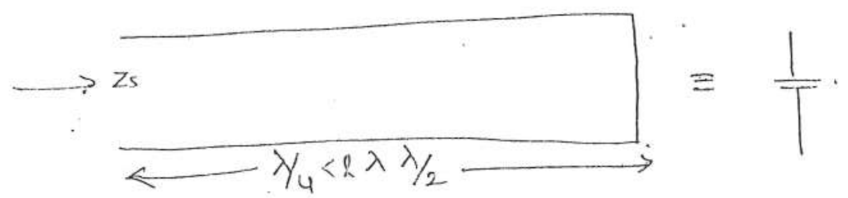
Transmission lines — Equivalent parameters.



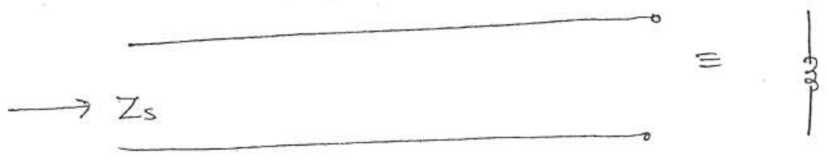
$Z_s = j Z_0 \tan\beta l$.



$Z_s = -j Z_0 \tan\beta l$

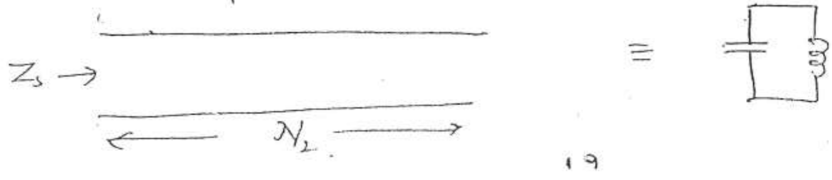


$Z_s = -j Z_0 \tan\beta l$



$Z_s = +j Z_0 \cot\beta l$.

$Z_s = \frac{2Z_0^2}{Rl}$.



$Z_s = \frac{2Z_0^2}{Rl}$

Input Impedance of a Loss less line : $(\alpha = 0)$

we Know $$Z_{in} = Z_0 \frac{Z_R + Z_0 \tanh pl}{Z_0 + Z_R \tanh pl}$$

$p = \alpha + j\beta$
lossless line $\alpha = 0$

For lossless line $\alpha = 0$ $\therefore$ $p = \alpha + j\beta = j\beta$

$$Z_{in} = Z_0 \frac{Z_R + Z_0 \tanh p l}{Z_0 + Z_R \tanh p l}$$

as $\tanh j\beta l = j \tan\beta l$

$$Z_{in} = Z_0 \frac{Z_R + jZ_0 \tan \beta l}{Z_0 + jZ_R \tan \beta l}$$

Since $\beta = \frac{2\pi}{\lambda}$

$$Z_{in} = Z_0 \frac{Z_R + jZ_0 \tan \frac{2\pi l}{\lambda}}{Z_0 + jZ_R \tan \frac{2\pi l}{\lambda}}$$

Input Impedance in terms of Reflection coefficient :-

$$Z_{in} = Z_0 \frac{Z_R \cosh pl + Z_0 \sinh pl}{Z_0 \cosh pl + Z_R \sinh pl}$$

$$Z_{in} = Z_0 \frac{Z_R\left(\frac{e^{pl} + e^{-pl}}{2}\right) + Z_0\left(\frac{e^{pl} - e^{-pl}}{2}\right)}{Z_0\left(\frac{e^{pl} + e^{-pl}}{2}\right) + Z_R\left(\frac{e^{pl} - e^{-pl}}{2}\right)}$$

Dividing Numerator & Denominator by $\frac{e^{pl}}{2}(Z_R + Z_0)$

$$Z_{in} = Z_0 \frac{1 + e^{-2pl}\left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)}{1 - e^{-2pl}\left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)}$$

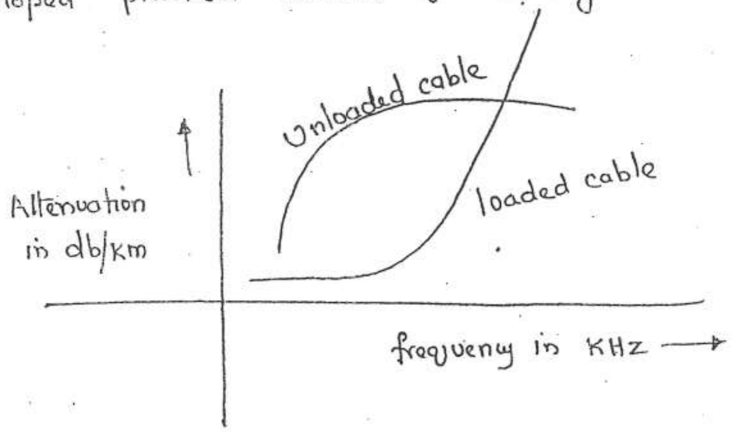
putting $K = \frac{Z_R - Z_0}{Z_R + Z_0}$

$$\therefore \boxed{Z_{in} = Z_0 \frac{1 + K e^{-2pl}}{1 - K e^{-2pl}}}$$

$$K = \frac{Z_R - Z_0}{Z_R + Z_0}$$

Loading in Transmission Lines :-

Increasing inductance by inserting inductance in series with the line is termed as loading and such lines are called as loaded lines. The theory of loading was developed by oliver Heaviside and Prof pupin developed practical method of loading.



For distortion less and for min attenuation conditions

we have $\frac{R}{L} = \frac{G}{C}$

The above conditions can be approached in 4 ways.

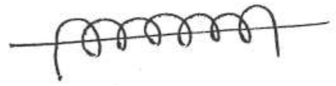
i) Reduce R : Decreases attenuation but require large conductors. cost increases.

ii) Increase G : It increases losses.

iii) Decrease C : This increases spacing between conductors, increases cable size, cost.

iv) Increase L : This decreases α and reduces distortion. Hence it is the best approach.

In practice, lumped inductors, known as loading coils are placed at suitable intervals along the line to increase the effective distributed inductance.

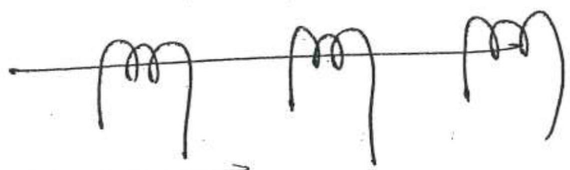


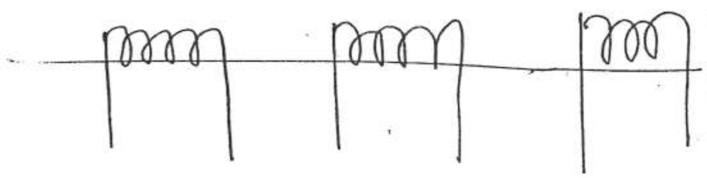
Different types of Loading :

i) continuous loading : In this a type of Iron or other magnetic material is wound round the conductor to be loaded. This method can increase the Inductance upto 65 mH per Km.

ii) patch loading :- This type of loading employs sections of continuously loaded cables seperated by section of unloaded cables. ex: submarine cable.

iii) Lumped loading : In this the inductance of a line can be increased by introduction of loading coil of oneform intervals.





Velocities in Transmission Lines :

Phase velocity (v_p) :- It is the velocity with which a wave of single frequency propagates along the line.

Group Velocity (v_g) :- It is the velocity at which enevelope of the complex wave propagates along the line.

Relation between V_g and V_p :-

we know that $V_p = \frac{\omega}{\beta}$

$$\frac{dv_p}{d\omega} = \frac{\beta \cdot - \omega \frac{d\beta}{d\omega}}{\beta^2} = \frac{1 - \frac{\omega}{\beta}\left(\frac{d\beta}{d\omega}\right)}{\beta}$$

$$\beta \frac{dv_p}{d\omega} = 1 - V_p \cdot \frac{1}{V_g}$$

$$\frac{V_p}{V_g} = 1 - \beta \frac{dv_p}{d\omega} \qquad \therefore V_g = \frac{V_p}{1 - \beta \frac{dv_p}{d\omega}}$$

$$\therefore V_g = \frac{V_p}{1 - \frac{\omega}{V_p} \frac{dv_p}{d\omega}}$$

If $\frac{dv_p}{d\omega} = 0$ then $\boxed{V_g = V_p}$

Computation of primary and secondary constants :-

$$Z_0 = \sqrt{\frac{R + J\omega L}{G + J\omega C}} \qquad P = \sqrt{(R + J\omega L)(G + J\omega C)}$$

$$(R + J\omega L) = Z_0 \times P \checkmark$$

$$(G + J\omega C) = \frac{P}{Z_0} \checkmark$$

---

and consists of many frequency components. They are of two types.

i) Frequeny distortion : It is due to different attenuations for different frequency components

2) Delay distortion :- It is that type of distortion in which the time required to transmit the various frequency components over the line and consequent delay is not constant. If V_p is independent of frequency, Delay distortion does not exist on lines.

3) phase distortion : It is due to different arrivals times of different components

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condition For distortion less Transmission :- $\left(\frac{R}{L} = \frac{G}{C}\right)$ ✓

Transmission is said to be distortionless if attenuation is indepeudent of frequency and phase shift is proportional to frequency.

we know $P = \sqrt{(R+J\omega L)(G+J\omega C)}$

$$= \sqrt{L\left(\frac{R}{L}+J\omega\right) C\left(\frac{G}{C}+J\omega\right)}$$

If $\frac{R}{L}+J\omega = \frac{G}{C}+J\omega$

$$P = \sqrt{LC}\left(\frac{R}{L}+J\omega\right) = R\sqrt{\frac{C}{L}} + J\omega\sqrt{LC}$$

$$= \sqrt{LC}\left(\frac{G}{C}+J\omega\right) = G\sqrt{\frac{L}{C}} + J\omega\sqrt{LC}$$

$$\alpha = R\sqrt{\frac{C}{L}} \text{ or } G\sqrt{\frac{L}{C}} \text{ or } \omega\sqrt{LC} = \beta$$

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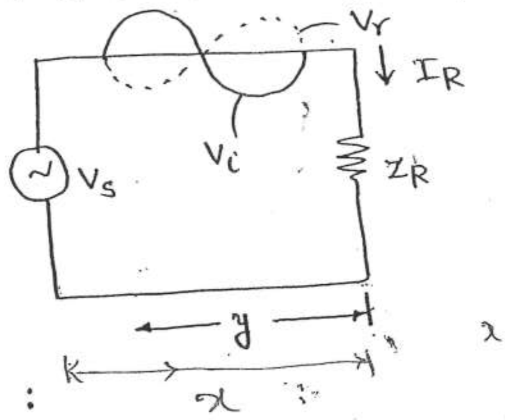
Reflection coefficient : (K) : It is defined as the ratio of the reflected voltage to incident voltage. It is a vector quantity having both magnitude and direction

$$K = \left|\frac{V_r}{V_i}\right| = \left|\frac{-I_r}{I_i}\right|$$



$x = -y$

K interms of Z_R and Z_0 :

The fondamental equations for the voltage and current are given by

$$V = a e^{Px} + b e^{-Px} \qquad e^{-Px} = \text{Incident voltage}$$

$$I = \frac{-1}{Z_0}\left(a e^{Px} + b e^{-Px}\right)$$

If Z_R is the terminating resistance and y is the distance measured from load then $x = -y$

$$V = b e^{Py} + a e^{-Py} \quad -(1)$$

$$I = \frac{b}{Z_0} e^{Py} - \frac{a}{Z_0} e^{-Py} \quad -(2)$$

Applying boundary condition: at $y = 0$ $V = V_R$, $I = I_R$

Substituting in 1 and 2 we have

$$V_R = b + a \qquad I_R = \frac{b}{Z_0} - \frac{a}{Z_0} = \frac{b-a}{Z_0}$$

$$I_R Z_0 = b - a$$

$$\therefore \quad 2b = V_R + I_R Z_0 \qquad b = \frac{V_R + I_R Z_0}{2}$$

$$a = \frac{V_R - I_R Z_0}{2}$$

By definition $K = \frac{V_r}{V_i} = \frac{a e^{-Py}}{b e^{Py}}$ at $y = 0$ $K = \frac{a}{b}$

$$K = \frac{V_R - I_R Z_0}{V_R + I_R Z_0} = \frac{Z_R - Z_0}{Z_R + Z_0}$$

$$\therefore \boxed{K = \frac{Z_R - Z_0}{Z_R + Z_0}}$$

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Problem :- A lossless line in Air has Z_0 of 350Ω and terminated by unknown impedance. when f is 1200MHZ, SWR is 9·48 and first voltage minima is situated at 6 cm from load. Determine the complex reflex coefficient and terminating impedance of the line.

Date : Given $Z_0 = 350$ $f = 1200\ MHZ$ $VSWR = 9{\cdot}48$ $Y_{min} = 6\times10^{-2}\,m$

wavelength $\lambda = \dfrac{c}{f} = \dfrac{3\times10^8}{1200\times10^6} = 0{\cdot}25\,m$ $\beta = \dfrac{2\pi}{\lambda} = 25{\cdot}12\ rad$

For voltage minima $2\beta Y_{min} - \phi = \pi$

$2\times 25{\cdot}12\times\dfrac{6}{100} - \phi = 3{\cdot}14$ $\phi = -0{\cdot}96\ rad$

$\therefore\ |K| = \dfrac{S-1}{S+1} = \dfrac{9{\cdot}48-1}{9{\cdot}48+1} = 0{\cdot}80916$

$K = |K|\,e^{j\phi} = 0{\cdot}80916\ e^{-j0{\cdot}96}$

$= 0.8096\,\angle 55^\circ.$

$K = \dfrac{Z_R - Z_0}{Z_R + Z_0} = 0{\cdot}81\,\angle 55^\circ = \dfrac{Z_R - 350}{Z_R + 350}$

$\therefore$ After evaluation

$Z_R = 659.74\,\angle -284{\cdot}58.$

Standing wave Ratio : It is the ratio of maximum and minimum magnitude of current or voltage on a standing wave is called standing wave Ratio. It is abbrevated as SWR or VSWR or S.

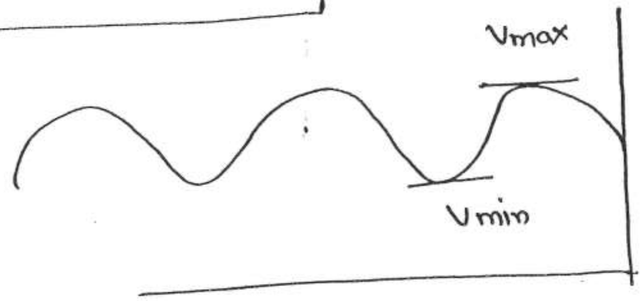
$$S = \frac{V_{max}}{V_{min}} = \frac{I_{max}}{I_{min}}$$

$$\therefore \ VSWR = \left|\frac{V_{max}}{V_{min}}\right| = \frac{|V_i|+|V_r|}{|V_i|-|V_r|} = \frac{1+\left|\frac{V_r}{V_i}\right|}{1-\left|\frac{V_r}{V_i}\right|} = \frac{1+|K|}{1-|K|}.$$

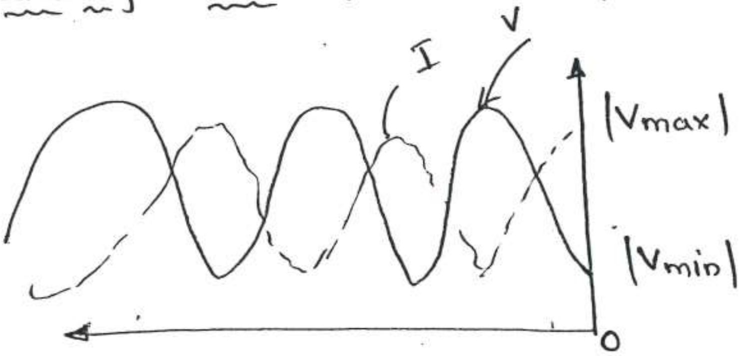
$$\boxed{\begin{aligned} \therefore \ VSWR = S &= \frac{1+K}{1-K} \\ K &= \frac{S-1}{S+1} \end{aligned}}$$

Range of S : 1 to ∞

Range of K : 0 to 1



Significance of Vmax and Vmin positions along the Tx line :-



Let the voltage on the line interms of V_s and V_r as

$$V = V_s e^{-j\beta z} + V_r e^{j\beta z}$$

For a time dependent wave

$$V = V_s e^{j\omega t} \cdot e^{-j\beta z} + V_r \cdot e^{j\omega t} e^{j\beta z}$$

$$\because \ \frac{V_s}{V_r} = K e^{-2\gamma l} \quad \text{and} \quad l - z = x .$$

Impedance at a Voltage minimum and voltage Maximum :

For a lossless line the impedance at voltage minimum and voltage maximum are purely real.

At a voltage maxima or current minima

$$Z_{in} = Z_{max} = \frac{V_{max}}{I_{min}}$$

$$= Z_0 \left(\frac{1+|K|}{1-|K|} \right) = Z_0 S$$

$$\therefore \boxed{Z_{max} = Z_0 S.}$$

At Voltage Minimum or current Maximum

$$Z_{in} = Z_{\cancel{max}} = \frac{V_{min}}{I_{max}}$$

$$= Z_0 \left(\frac{1-|K|}{1+|K|} \right) = \frac{Z_0}{S}$$

$$\therefore \boxed{Z_{min} = \frac{Z_0}{S}}$$

Applications of Transmission Lines :-

1) Impedance Transformer (Impedance Inverter)
2) Single stub Matching
3) Double stub Matching
4) VSWR measurement.
5) Unknown Impedance measurement.

Relation between SWR & Relection coefficient :—

$$|V_{max}| = |V_i| + |V_r|$$

$$|V_{min}| = |V_i| - |V_r|$$

$$VSWR = \left|\frac{V_{max}}{V_{min}}\right| = \frac{1 + \left|\frac{V_r}{V_i}\right|}{1 - \left|\frac{V_r}{V_i}\right|} = \frac{1+K}{1-K}$$

$$\boxed{\therefore \; K = \frac{S-1}{S+1} = \frac{Z_R - Z_0}{Z_R + Z_0}}$$

Impedance Matching Devices :-

To achieve Max power Transfer we use

i) Quarter wave Transformer :

$$Z_{in} = Z_0 \frac{Z_R \cosh pl + Z_0 \sinh pl}{Z_0 \cosh pl + Z_R \sinh pl} \qquad l = \lambda/4$$

$$\beta = 2\pi/\lambda$$

For lossless $(\alpha = 0)$

$$Z_{in} = \frac{Z_R \cos\beta l + J Z_0 \sin\beta l}{Z_0 \cos\beta l + J Z_R \sin\beta l}$$

for $l = \lambda/4$ $\quad \beta = 2\pi/\lambda$

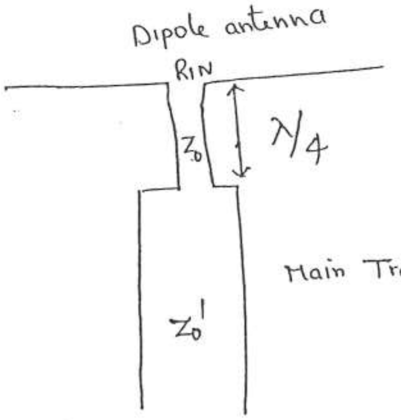
$$Z_{in} = Z_0 \frac{Z_R \cos \pi/2 + J Z_0 \sin \pi/2}{Z_0 \cos \pi/2 + J Z_R \sin \pi/2}$$

$$= Z_0 \frac{Z_0}{Z_R} = \frac{Z_0^2}{Z_R}$$

$$\therefore \; Z_0 = \sqrt{Z_{in} Z_R}$$

∴ The product of input impedance and load impedance equal to square of characteristic impedance.

Application of Quarter wave Transformer :-



Characteristic impedance of Quarter wave Transformer is

$$Z_0 = \sqrt{Z_0' R_{IN}}$$

why it is an impedance inverter :-

we know that $Z_0 = \sqrt{Z_{in} Z_R}$

If Z_R is high Z_{in} is low
If Z_R is low Z_{in} is high
} Assuming Z_0 is resistive

If the load is inductive, Z_{in} is capacitive & vice versa.

Depending on value of Z_R, Quarter wave acts as step up or step down impedance Transformer.

Disadvantage : It is sensitive to change in frequency. If the freq changes its length no longer will be $\lambda/4$.

$$l = \lambda/4 \qquad \lambda = \frac{c}{f} = \frac{3\times10^8}{f}$$

Problems :

Given that

Open-circuited impedance, $Z_{oc} = 750\ \Omega$.

Short-circuited impedance, $Z_{sc} = 500\ \Omega$.

Find characteristic Impedance (Z_0) :

The characteristic impedance of a line is given by

$$Z_0 = \sqrt{Z_{oc} \times Z_{sc}} = \sqrt{750 \times 500} = \sqrt{375000}.$$

The characteristic impedance is $Z_0 = 612.37\ \Omega$.

Example 10.13 $Z_{oc} = 900\angle{-30°}\Omega$, $Z_{sc} = 400\angle{-10°}\Omega$. Calculate the Z_0 and γ of a 12 km long line.

Solution Given: $Z_{oc} = 900\angle{-30°}\Omega$, $Z_{sc} = 400\angle{-10°}\Omega$.

a) The characteristic impedance is given by

$$Z_0 = \sqrt{Z_{oc} \times Z_{sc}} = \sqrt{900\angle{-30} \times 400\angle{-10}} = 600\angle{-20°}\ \Omega.$$

Hence, the characteristic impedance, $Z_0 = 600\angle{-20°}\ \Omega$.

We know that $\tanh \gamma l = \sqrt{\dfrac{Z_{sc}}{Z_{oc}}}$.

$$\tanh \gamma l = \sqrt{\frac{400\angle{-10}}{900\angle{-30}}} = 0.67\angle 10.$$

$$\frac{e^{\gamma l} - e^{-\gamma l}}{e^{\gamma l} + e^{-\gamma l}} = 0.66 - j0.116.$$

By componendo and dividendo principle,

$$\frac{2e^{\gamma l}}{2e^{-\gamma l}} = \frac{1 + 0.66 - j0.116}{1 - (0.66 - j0.116)}.$$

$$e^{2\gamma l} = 4.269 - j1.798 = 4.632\angle{-22.8}.$$

Taking ln of both sides,

$$2\gamma l = \ln(4.632\angle{-22.8°}).$$

$$\gamma = \frac{1}{2l}\left(\ln(4.632) + j(-22.8°)\right)$$

or $$\gamma = \frac{1}{2 \times 12}(1.533 - j0.398).$$

$$\gamma = \alpha + j\beta = 0.0638 - j0.01658 \text{ napers/km}.$$

Example 10.14 A two wire line has a characteristic impedance of 300 Ω and is fed to a 90 Ω resistor at 200 MHz. A quarter wave line is used as a tube 0.25 cm in diameter. Find the centre-to-centre spacing in air.

Solution Given: $Z_1 = 300\ \Omega$, $Z_2 = 90\ \Omega$, tube diameter $= 0.25$ cm, radius $r = 0.125$ cm.

For a quarter wave line, we know that

$$Z_0 = \sqrt{Z_1 \times Z_2} = \sqrt{300 \times 90} = 164.32\ \Omega.$$

Also, we know that the characteristic impedance of a parallel wire line is,

$$Z_0 = 276 \log (d/r) \text{ ohms},$$

where $d =$ centre-to-centre spacing.

$$\therefore \quad 164.32 = 276 \log \left[\frac{d}{0.125}\right].$$

$$\log \left[\frac{d}{0.125}\right] = 0.595.$$

$$\frac{d}{0.125} = 3.936 \quad \text{or} \quad d = 0.49 \text{ cm}.$$

Example 10.15 A 60 ohm lossless line is connected to a source with 10 V, $Z_g = 50 - j40$ and terminated with a load of $j40$ ohms. If the line is 100 m long and $\beta = 0.25$ rad/m, calculate Z_{in} and voltage at (i) the sending end, (ii) the receiving end, (iii) 4 m from the load end and (iv) 3 m from the source.

Solution Given:
Length, $l = 100$ m, characteristic impedance, $Z_0 = 60$ ohms, termination impedance, $Z_R = j40$ ohms, source resistance, $Z_g = 50 - j40$ ohms, source voltage, $V_g = 10$ V, $\alpha = 0$, $\beta = 0.25$ rad/m,

$$\text{input current is } I_s = \frac{V_g}{Z_0 + Z_g} = \frac{10}{60 + 50 - j40} = 85.4\angle 20 \text{ mA}.$$

We know that the source impedance for a lossless transmission line is

$$Z_s = Z_0 \left(\frac{Z_R + jZ_0 \tan \beta l}{Z_0 + jZ_R \tan \beta l}\right).$$

(i) At the sending end, $l = 100$ m, $\beta l = 0.25 \times 100 = 25$ rad $= 1432.4°$.

$$Z_s = 60 \left(\frac{j40 + j60 \tan 1432.4}{60 + j^2 40 \tan 1432.4}\right) = j29.38\ \Omega.$$

Input voltage is $V_s = Z_s I_s = j29.38 \times (85.4\angle 20) \times 10^{-3} = 2.5\angle 110$ V.

(ii) At the receiving end, $l = 0$.

$$Z_s = Z_R = j40\ \Omega.$$

$$V_R = Z_R I_s = j40 \times (85.4\angle 20) \times 10^{-3} = 3.416\angle 110 \text{ V}.$$

(iii) At 4 m from the load end: $l = 4$, $\beta l = 0.25 \times 4 = 1$ rad $= 57.3°$.

$$Z_1 = 60\left(\frac{j40 + j60\tan 57.3}{60 + j^2 40\tan 57.3}\right) = -j435.53\ \Omega.$$

$$V_1 = Z_1 I_s = -j435.53 \times 85.4\angle 20 \times 10^{-3} = 37.2\angle -70° \text{ V}.$$

(iv) At 3 m from the source end: $l = 97$, $\beta l = 025 \times 97 = 2425$ rad $= 1389.4°$.

$$Z_2 = 60\left(\frac{j40 + j60\tan 1389.4}{60 + j^2 40\tan 1389.4}\right) = -j0.303\ \Omega.$$

$$V_2 = Z_2 I_s = -j0.303 \times (85.4\angle 20) \times 10^{-3} = 0.0258\angle -70° \text{ V}.$$

Example 10.16 An open wire unloaded line, 75 km long, is operated at a frequency of 1000 Hz. The open circuit impedance is found to be $330\angle -30°\ \Omega$ and the short circuit impedance is $540\angle 7°\ \Omega$. Calculate the parameters of line.

Solution Given: Length of the unloaded line, $l = 75$ km, $f = 1000$ Hz, $Z_{oc} = 330\angle -30°\ \Omega$, $Z_{sc} = 540\angle 7°\ \Omega$.

We know that $Z_0 = Z_{sc} \times Z_{oc}$

$$= \sqrt{540\angle 7° \times 330\angle -30°} = 422.14\angle -11.5.$$

Also, $Z_{sc} = Z_0 \tanh \gamma l$.

$$\tanh \gamma l = \frac{Z_{sc}}{Z_0} = \frac{540\angle 7°}{422.14\angle -23°}.$$

$$= 1.28\angle 30° = 1.108 + j0.64.$$

$$\frac{e^{\gamma l} - e^{-\gamma l}}{e^{\gamma l} + e^{-\gamma l}} = 1.108 + j0.64.$$

By componendo and dividendo principle,

$$\frac{2e^{\gamma l}}{2e^{-\gamma l}} = \frac{1 + 1.108 - j0.64}{1 - 1.108 - j0.64}.$$

$$e^{2\gamma l} = \frac{2.108 + j0.64}{-0.108 - j0.64}.$$

$$e^{2\gamma l} = \frac{2.203\angle 16.89°}{0.649\angle -99.58°} = 3.394\angle 116.47°.$$

Taking ln of both sides,

$$2\gamma l = \ln(3.394\angle 116.47^\circ).$$

Since, $\ln(r\angle\theta) = \ln r + j\theta^\circ$,

$$\gamma = \frac{1}{2l}(\ln(3.394) + j(116.47^\circ))$$

$$= \frac{1}{2l}(1.222 + j2.033) = \frac{1.222 + j2.033}{150} = 0.00815 + j0.0135.$$

$$\gamma = 0.0157\angle 58.88^\circ.$$

$$\beta = 0.00815 \text{ rad/km} \quad \text{and} \quad \alpha = 0.0135 \text{ nepers/km}.$$

Also we know that $= R + j\omega L$.

$$R + j\omega L = 0.0157\angle 58.88^\circ \times 422.14\angle -10.5^\circ = 6.627\angle 47.38$$

$$R + j\omega L = 4.487 + j4.877.$$

$$R = 4.487\ \Omega/\text{km} \quad \text{and} \quad \omega L = 4.877.$$

$$2\pi f \times L = 4.877.$$

$$L = \frac{4.877}{2\pi f} = \frac{4.877}{2 \times \pi \times 1000} = 0.776 \text{ mH/km}.$$

$$\frac{\gamma}{Z_0} = G + j\omega C.$$

$$G + j\omega C = \frac{0.0157\angle 58.88^\circ}{422.14\angle -10.5^\circ} = 3.72 \times 10^{-5}\angle 70.38^\circ$$

$$G + j\omega C = 12.5 \times 10^{-6} + j35 \times 10^{-6}.$$

$$G = 12.5\ \mu\mho/\text{km and} \quad \omega C = 3.5 \times 10^{-5}.$$

$$2\pi f C = 3.5 \times 10^{-5}.$$

$$C = \frac{3.5 \times 10^{-5}}{2\pi f} = \frac{3.5 \times 10^{-5}}{2 \times \pi \times 1000}$$

$\therefore$ $$C = 5.576 \text{ nF/km}.$$

Example 10.17 The input impedance of a short circuited lossy transmission line of length 2 m and characteristic impedance 75 Ω is $45 + j225$ Ω.

(a) Find α and β of a line.
(b) Determine the input impedance if the short circuit is replaced by a load impedance of $67.5 - j45$ Ω.

Solution Given: Input impedance of a short circuited line, $Z_{sc} = 45 + j225\ \Omega$, characteristic impedance, $Z_0 = 75\ \Omega$, length $l = 2$ m.

(a) We know that $Z_{sc} = Z_0 \tanh \gamma l$.

$$\tanh \gamma l = \frac{Z_{sc}}{Z_0} = \frac{45 + j225}{75} = 0.6 + j3.$$

$$\frac{e^{\gamma l} - e^{-\gamma l}}{e^{\gamma l} + e^{-\gamma l}} = 0.6 + j3.$$

By componendo and dividendo principle,

$$\frac{2e^{\gamma l}}{2e^{-\gamma l}} = \frac{1 + 0.6 - j3}{1 - 0.6 - j3}.$$

$$e^{2\gamma l} = \frac{1.6 + j3}{0.4 - j3} = 1.12\angle 114.3.$$

Taking ln of both sides,

$$2\gamma l = \ln(1.12\angle 144.3^\circ).$$

Since, $\ln(r\angle\theta) = \ln r + j\theta^\circ$,

$$\gamma = \frac{1}{2l}(\ln(1.12) + j(144.3^\circ)) = \frac{0.1133 + j2.518}{2 \times 2} = 0.028 + j0.63.$$

$$\gamma = 0.0157\angle 58.88^\circ;\ \beta = 0.63 \text{ rad/km and } \alpha = 0.028 \text{ nepers/km}.$$

(b) Given: Load impedance $Z_R = 67.5 - j45\ \Omega$.
We know that the input impedance is

$$Z_{\text{in}} = Z_0 \left(\frac{1 + Ke^{-2\gamma l}}{1 - Ke^{-2\gamma l}} \right),$$

where $K = \dfrac{Z_R - Z_0}{Z_R + Z_0} = \dfrac{67.5 - j45 - 75}{67.5 - j45 + 75} = 0.3\angle -82^\circ.$

$$e^{2\gamma l} = 1.12\angle 114.3.$$

$$Z_{\text{in}} = 75 \left(\frac{1 + \dfrac{0.3\angle -82}{1.12\angle 144.3}}{1 - \dfrac{0.3\angle -82}{1.12\angle 144.3}} \right) = 75 \left(\frac{1 + 0.2678\angle 226.3}{1 - 0.2678\angle 226.3} \right) = 52.35\angle 22.66.$$

Input impedance is $Z_{\text{in}} = 48.3 + j20\ \Omega$.

Example 10.18 A dipole antenna is fed by a lossless transmission line having $Z_o = 60\ \Omega$. The source impedance is 600 Ω. If the length of the line is 0.1λ, determine antenna impedance.

Solution Given: Length of the line $l = 0.1\lambda$, characteristic impedance, $Z_0 = 60\ \Omega$, source impedance $Z_S = 600\ \Omega$.

Let Z_R be the antenna impedance. We know that the source impedance for a lossless transmission line is

$$Z_S = Z_0\left(\frac{Z_R + jZ_0 \tan \beta l}{Z_0 + jZ_R \tan \beta l}\right),$$

or $$Z_R = Z_0\left(\frac{Z_S - jZ_0 \tan \beta l}{Z_0 - jZ_S \tan \beta l}\right).$$

Now $$\tan \beta l = \tan\left[\frac{2\pi}{\lambda}(0.1\lambda)\right] = \tan(0.2\pi) = 0.726.$$

$$\therefore \quad Z_R = 60\left[\frac{600 - j60(0.726)}{60 - j600(0.726)}\right] = 60\left[\frac{600 - j43.56}{60 - j435.6)}\right]$$

$$= 17.06 + j80.3 = 82.09\angle 78°.$$

$\therefore$ Antenna impedance is $Z_R = 82.09\angle 78°$.

Example 10.19 A transmission line of 100 Ω characteristic impedance is connected to a load of 400 Ω. Calculate the reflection coefficient and standing wave ratio.

Solution Given: Characteristic impedance, Z_0= 100 Ω, load, $Z_R = 400\ \Omega$.

We know that reflection coefficient K is given by

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{400 - 100}{400 + 100}.$$

$\therefore$ Reflection coefficient, $K = 3/5 = 0.6$.

The standing wave ratio is $\text{VSWR} = \dfrac{1+|K|}{1-|K|} = \dfrac{1+0.6}{1-0.6} = \dfrac{1.6}{0.4} = 4.$

Example 10.20 An UHF transmission line of $Z_0 = 150$ ohms is terminated with an unknown load. The VSWR measured in the line is 5 and the position of minimum current nearest the load is one-fifth wavelength away. Calculate the value of the load impedance.

Solution Given: Characteristic impedance, $Z_0 = 150$, VSWR $= S = 5$, position of minimum current $y_{\max} = \lambda/5$.

Position of the first minimum current or voltage maximum can be obtained by

$$2\beta y_{\max} - \phi = 0.$$

$$2.\frac{2\pi}{\lambda}.\frac{\lambda}{5} - \phi = 0$$

or $\phi = \dfrac{4\pi}{5} = 2.51$ radians or $144°$.

We know that the magnitude of the reflection coefficient is

$$|K| = \frac{S-1}{S+1} = \frac{5-1}{5+1} = 0.666.$$

The reflection coefficient K is

$$K = |K|e^{j\phi} = 0.667\angle 144°.$$

Also we know that $K = \dfrac{Z_R - Z_0}{Z_R + Z_0}$.

$$0.667\angle 144° = \frac{Z_R - 150}{Z_R + 150}$$

$$-0.5388 + j0.3916 = \frac{Z_R - 150}{Z_R + 150}$$

$$Z_R(-0.5388 + j0.3916 - 1) = (-150 + 150 \times 0.5388 - j150 \times 0.3916)$$

$$Z_R(-1.5388 + j0.3916) = -69.18 - j58.74$$

$$Z_R = \frac{90.753\angle -139.66}{1.588\angle 165.72}.$$

The load impedance, $Z_R = 57.15\angle -305.38$ or $Z_R = 57.15\angle 54.62\ \Omega$.

Example 10.21 An open wire transmission line having $Z_0 = 650\angle -12°\ \Omega$ is terminated in Z_0 at the receiving end. If this line is supplied from a source of internal resistance 300 Ω, calculate the reflection factor and reflection loss at the sending end terminals.

Solution Given: $Z_i = 300\ \Omega$, $Z_0 = 650\angle -12°\ \Omega$.
We know that the reflection factor is

$$K_f = \frac{2\sqrt{Z_i Z_0}}{(Z_i + Z_0)}.$$

$$\therefore \quad K_f = \frac{2\sqrt{300 \times 650\angle -12°}}{(300 + 6358 - j135)} = \frac{883\angle -6}{945.5\angle -8.22} = 0.934\angle 2.22°.$$

The reflection loss,

$$K_e = 20\log_{10}\left|\frac{1}{K_f}\right| = 20\log\left(\frac{1}{0.934}\right).$$

$$K_e = 0.59 \text{ dB}.$$

$\therefore$ The reflection loss is 0.59 dB.

Example 10.22 A 80 Ω distortionless line connects a signal of 50 kHz to a load of 140 Ω. The load power is 75 mW. Calculate:

(i) Voltage reflection coefficient,

(ii) VSWR,

(iii) Position of V_{max}, I_{max}, V_{min} and I_{min}.

Solution Given: $Z_R = 140\ \Omega$, $Z_0 = 80\ \Omega$.

(i) Voltage reflection coefficient

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{140-80}{140+80} = 0.273.$$

$$\therefore \quad K = 0.273, \quad \phi = 0.$$

(ii) VSWR

$$\text{VSWR} = \frac{1+|K|}{1-|K|} = \frac{1+0.273}{1-0.273} = 1.75.$$

(iii) Position of V_{max}, I_{max}, V_{min} and I_{min}

The condition for maximum voltage occurs at y_{max} from the load, i.e.,

$$2\beta y_{max} - \phi = 2n\pi.$$

For first maximum, $n = 0$,

$$2\beta y_{max} - \phi = 0$$

$$2\beta y_{max} - 0 = 0, \quad y_{max} = 0.$$

Therefore, the first voltage maximum, V_{max}, and current minimum, I_{min}, occur at the load position.

The first voltage minimum, V_{min}, occurs at a distance of $\lambda/4$ from V_{max}.

$$\therefore \quad y_{min} = y_{max} + \lambda/4 = 0 + \lambda/4 = \lambda/4.$$

We know that the wavelength, $\lambda = \dfrac{v_0}{f} = \dfrac{3\times 10^8}{50\times 10^3} = 6$ km.

$$\therefore \quad y_{min} = \frac{v_0}{4\times f} = \frac{3\times 10^8}{4\times 50\times 10^3} = 1.5 \text{ km}.$$

Therefore, the first voltage minimum and current maximum, I_{max} occur at a distance of 1.5 km from the load. These values repeat every $\dfrac{\lambda}{2} = 3$ km distance from the load.

Given: The power at the load is $P = 75$ mW.

We know that $P = \dfrac{V_{max}^2}{ZR}$ and $I_{min} = \dfrac{V_{max}}{Z_R}$.

V_{max} at the load is $\sqrt{P \times Z_R} = \sqrt{75 \times 10^{-3} \times 140} = 3.24$ volts and $I_{min} = \dfrac{3.24}{140} = 0.23$ mA.

Example 10.23 Design a quarter wave transformer to match a line having impedance of 300 Ω to load of 600 Ω.

Solution Given: $Z_0 = 300\ \Omega$, $Z_L = 600\ \Omega$.

The quarter wave transformer should have a sending end impedance of

$$Z_S = \frac{Z_0^2}{Z_L} = \frac{300 \times 300}{600} = 150\ \Omega.$$

The impedance will be matched if the Z_S of a $\lambda/4$ transformer is 150 Ω.

Example 10.24 Calculate the reflection coefficient and VSWR of a 50 Ω line terminated with (i) matched load, (ii) short circuit, (iii) $+j50\ \Omega$ load, (iv) $-j50\ \Omega$ load.

Solution

(i) Matched load

$$Z_0 = 50\ \Omega, \quad Z_R = 50\ \Omega.$$

Reflection coefficient $K = \dfrac{Z_R - Z_0}{Z_R + Z_0} = \dfrac{50-50}{50+50} = 0$

and VSWR $S = \dfrac{1+|K|}{1-|K|} = \dfrac{1+0}{1-0}$.

$\therefore \quad S = 1$

(ii) Short circuit

$$Z_R = 0,\ Z_0 = 50\ \Omega.$$

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{0-50}{0+50} = -1.$$

$$S = \frac{1+|K|}{1-|K|} = \frac{1}{0} = \infty.$$

(iii) $+j50\ \Omega$ load

$\therefore \quad Z_R = +j50\ \Omega,\ Z_0 = 50\ \Omega.$

$$K = \frac{+j50 - 50}{j50 + 50} = \frac{-1+j}{1+j} = \frac{1.414\angle 135^\circ}{1.414\angle 45^\circ}$$

$\therefore \quad K = 1\angle 90^\circ.$

$$S = \frac{1+|K|}{1-|K|} = \frac{1+1}{1-1} = \frac{2}{0} = \infty.$$

(iv) $-j50\ \Omega$ load

$\therefore \quad Z_R = -j50\ \Omega,\ Z_0 = 50\ \Omega.$

$$K = \frac{-j50-50}{j50+50} = \frac{1.414\angle -135^\circ}{1.414\angle -45^\circ} = 1\angle -90^\circ.$$

$$\therefore \quad S = \frac{1+1}{1-1} = \frac{2}{0} = \infty.$$

Example 10.25 An aerial of $(200 - j300)\ \Omega$ is to be matched with $500\ \Omega$ lines. The matching is to be done by means of a low loss $600\ \Omega$ stub line. Find the position and length of the stub line used if the operating wave length is 20 metres.

Solution Given: $Z_R = 200 - j300\ \Omega = 360.55\angle -56.31$, $Z_0 = 500\ \Omega$, $\lambda = 20$ m.

We know that the reflection coefficient is

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{200 - j300 - 500}{200 - j300 + 500} = \frac{-300 - j300}{700 - j300} = 0.2068 - j0.5172.$$

$\therefore \quad K = 0.557\angle -110.8.$

So, $|K| = 0.557$ and $\phi = -110.8^\circ$.

We know that the position of the stub line l_s in terms of the reflection coefficient is

$$l_s = \frac{\lambda}{2\pi}(\phi + \pi - \cos^{-1}(|K|).$$

Substituting the values of λ, ϕ and $|K|$, we get,

$$l_s = \frac{20}{2\pi}(-110.8 + \pi - \cos^{-1}(0.557))$$

$$= \frac{10}{180}(-110.8 + 180 - 56.15) = \frac{10 \times 12.05}{180} = 0.669.$$

The stub position from the aerial is

$$l_s = 0.669 \text{ metres.}$$

Also the length of the stub line l_s in terms of the reflection coefficient $|K|$ is

$$l_t = \frac{\lambda}{2\pi}\tan^{-1}\frac{\sqrt{1-|K|^2}}{2|K|}.$$

Stub Matching

Stub : It is a piece of a Transmission line. Basically there are two types of stubs. i) open circuited stub 2) short circuited stub. short circuited stubs are preferable as they are less likely to radiate.

Single stub Matching :

Advantages : i) The length and characteristic Impedance need not be changed.

2) Radiation losses are less.

Disadvantages : 1) Single stub system is not useful for variable frequency. If frequency changes location of the stub has to be changed.

2) If the terminating Impedance changes the position and length of the stub is to be changed

Double stub Matching :-

Advantages : 1) It is frequency independent.

2) Reduces the reflections on the lines.

Designing : 1) Design is complex.

2). Radiation is more

3) costly

4) Allignment is difficult

5) stubs are more.

Problem on stub Matching :- An aerial of $200-j300\,\Omega$ is to be matched with 500Ω lines. The Matching is to be done by means of low loss 600Ω stub line. Find the position and length of the stub if the operating wave length is 20 metres.

Answer : $Z_R = 200-j300 = 360.55\angle{-56.30}$, $Z_0 = 500\Omega$ $\lambda = 20\,M$

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = 0.557\angle{-111.8}$$

$$|K| = 0.557$$

$$\phi = -111.8^\circ = -0.6211\pi$$

$$\boxed{l_s = \frac{\lambda}{2\pi}\left(\phi + \pi - \cos^{-1}(|K|)\right)}$$

$$l_s = \frac{20}{2\pi}\left(-0.6211\pi + \pi - \cos^{-1}(0.557)\right)$$

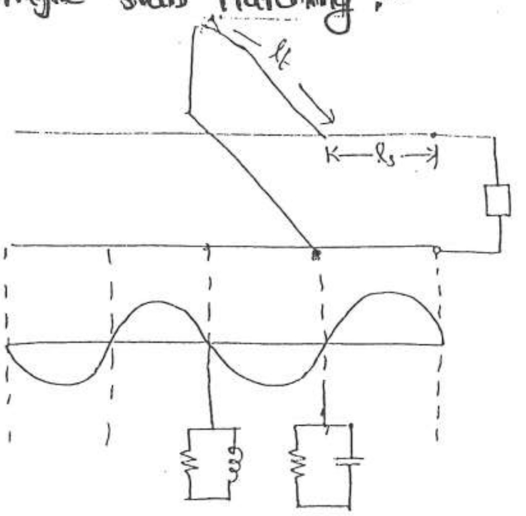
$$l_s = 0.6693 \text{ metres.}$$

$$\boxed{l_t = \frac{\lambda}{2\pi}\tan^{-1}\frac{\sqrt{1-|K|^2}}{2|K|}}$$

$$l_t = \frac{20}{2\pi}\tan^{-1}\frac{\sqrt{1-(0.557)^2}}{2(0.557)}$$

$$\boxed{l_t = 2.039 \text{ metres}}$$

Single stub Matching :-



$$Z_{in} = Z_0 \frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l}$$

$$Y_{in} = Y_0 \frac{Y_R + Y_0 \tanh \gamma l}{Y_0 + Y_R \tanh \gamma l}.$$

$$Y_{in} = \frac{Y_r + J \tan \beta l}{1 + J Y_r \tan \beta l} \cdot \frac{(1 - J Y_r \tan \beta l)}{(1 - J Y_r \tan \beta l)}$$

The stub has to be located where real part = 1.

$$\frac{Y_r(1 + \tan^2 \beta l_s)}{1 + Y_r^2 \tan^2 \beta l_s} = 1$$

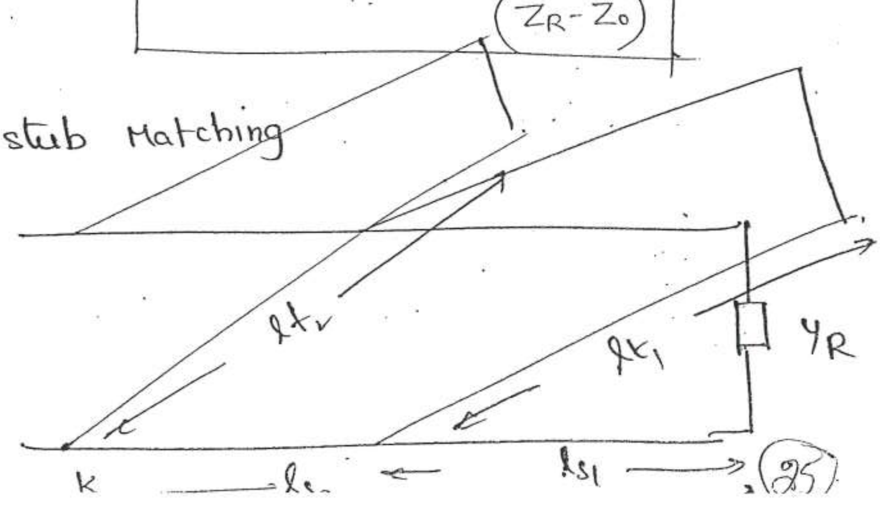
$$\therefore \boxed{l_s = \frac{\lambda}{2\pi} \tan^{-1} \sqrt{\frac{Z_R}{Z_0}}}$$

Advantages of short circuited stub :- i) radiates less power
ii) its effective length may be varied

The length of the stub l_t is

$$\boxed{l_t = \frac{\lambda}{2\pi} \tan^{-1} \frac{\sqrt{Z_R Z_0}}{(Z_R - Z_0)}}$$

Double stub Matching



The normalised admittance at stub 2 should be equal to

$$y_B = 1 \pm jb_2.$$

5. Connect a short-circuited stub 2 having susceptance $\mp jb_2$ to the line at 2.
6. The stub lengths can be obtained from Eq. (10.103);

$$b_1 = \cot \beta l_{t_1}, \qquad b_2 = \cot \beta l_{t_2}. \tag{10.112}$$

10.17 Smith Chart

Phillip H. Smith in the year 1939 developed a polar chart for calculating transmission line characteristics. This chart is called Smith chart. It consists of two sets of orthogonal circles which represent the values of normalised impedance. One set of circles represents the resistive component R, called R circles, and the other set of circles represent the reactive component X, called X circles.

Derivation of R circles and X circles

Consider a transmission line having characteristic impedance Z_0 terminated with Z_R. The reflection coefficient is

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{\dfrac{Z_R}{Z_0} - 1}{\dfrac{Z_R}{Z_0} + 1} = \frac{z_R - 1}{z_R + 1},$$

where $z_R = \dfrac{Z_R}{Z_0}$ is the normalised load impedance.

$$z_R = \frac{1+K}{1-K}.$$

Since z_R and K are complex quantities,

$$R + jX = \frac{1 + K_R + jK_X}{1 - K_R - jK_X},$$

where R and X are the real and imaginary parts of z_R, K_R and K_X are the real and imaginary parts of K respectively. Separating the real and imaginary terms,

$$R + jX = \frac{(1 + K_R + jK_X)(1 - K_R + jK_X)}{(1 - K_R - jK_X)(1 - K_R + jK_X)}$$

$$= \frac{1 - K_R + jK_X + K_R - K_R^2 + jK_RK_X + jK_X - jK_RK_X - K_X^2}{(1 - K_R)^2 + K^2}.$$

$$R + jX = \frac{1 - K_R^2 - K_X^2 + j2K_X}{(1 - K_R)^2 + K_X^2}. \tag{10.113}$$

$$\therefore \qquad R = \frac{1 - K_R^2 - K_X^2}{(1 - K_R)^2 + K_X^2} \tag{10.114}$$

and $$X = \frac{2K_X}{(1-K_R)^2 + K_X^2}. \tag{10.115}$$

Arranging the above equations in standard form, the real part is

$$R = \frac{1-K_R^2-K_X^2}{(1-K_R)^2+K_X^2}.$$

$$R(1-K_R)^2 + RK_X^2 = 1 - K_R^2 - K_X^2$$

$$R\left(1+K_R^2-2K_R\right) + RK_X^2 = 1-K_R^2-K_X^2$$

$$(1+R)K_X^2 + (1+R)K_R^2 + R - 2RK_R = 1.$$

$$K_R^2 + K_X^2 - \frac{2R}{(1+R)^2}K_R = \frac{1-R}{1+R}.$$

$$\left(K_R - \frac{R}{1+R}\right)^2 - \frac{R^2}{(1+R)^2} + K_X^2 = \frac{1-R}{1+R}$$

$$\left(K_R - \frac{R}{1+R}\right)^2 + K_X^2 = \frac{1-R}{1+R} + \frac{R^2}{(1+R)^2}$$

$$\left(K_R - \frac{R}{1+R}\right)^2 + K_X^2 = \frac{1}{(1+R)^2}. \tag{10.116}$$

This equation represents a family of R circles on the K plane (reflection coefficient plane) as shown in Fig. 10.18.

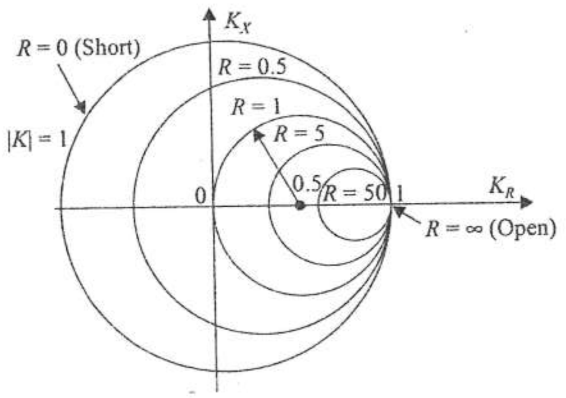


Fig. 10.18 Family of constant R circles

These circles are called constant R circles, having centres at $(R/(1+R), 0)$ and radii of $(1/(1+R))$. A set of circles can be generated at different values of R.

At $R = 0$, the centre of the circle is at (0, 0) and the radius is 1. This is an outer circle. As R increases, the circle radius decreases. $R = \infty$, represents a point at (1, 0). All circles touch the point (1, 0). Now the imaginary part is

$$X = \frac{2K_X}{(1-K_R)^2 + K_X^2}.$$

$$(1-K_R)^2 + K_X^2 = \frac{2K_X}{X}$$

$$(1-K_R)^2 + K_X^2 - \frac{2K_X}{X} = 0$$

$$(K_R - 1) + \left(K_X - \frac{1}{X}\right)^2 - \frac{1}{X^2} = 0$$

$$(K_R - 1)^2 + \left(K_X - \frac{1}{X}\right)^2 - \frac{1}{X^2} = 0. \tag{10.117}$$

This equation represents another family of circles on the K-plane as shown in Fig. 10.19.

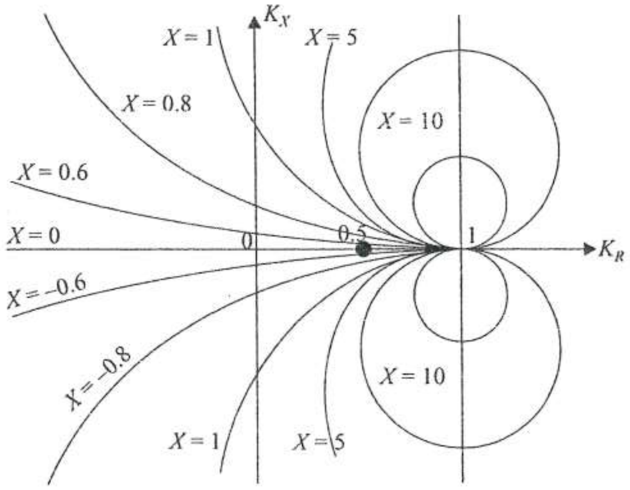


Fig. 10.19 *Family of constant X circles*

These circles are called constant X circles, having centres at $(1, 1/X)$ and radii of $1/X$.

$X = 0$ represents a straight line along the K_R-axis. If X is positive, then K_X is positive and the circles are generated above the $K_R = 0$ axis. If X is negative, then K_X is negative and the circles are generated below the $K_R = 0$ axis. $X = \infty$ represents a point at (1, 0). All circles touch the point (1, 0).

The complete Smith chart can be obtained by the superposition of the family of the constant-R circles and constant- X circles on the K plane as shown in Fig. 10.20.

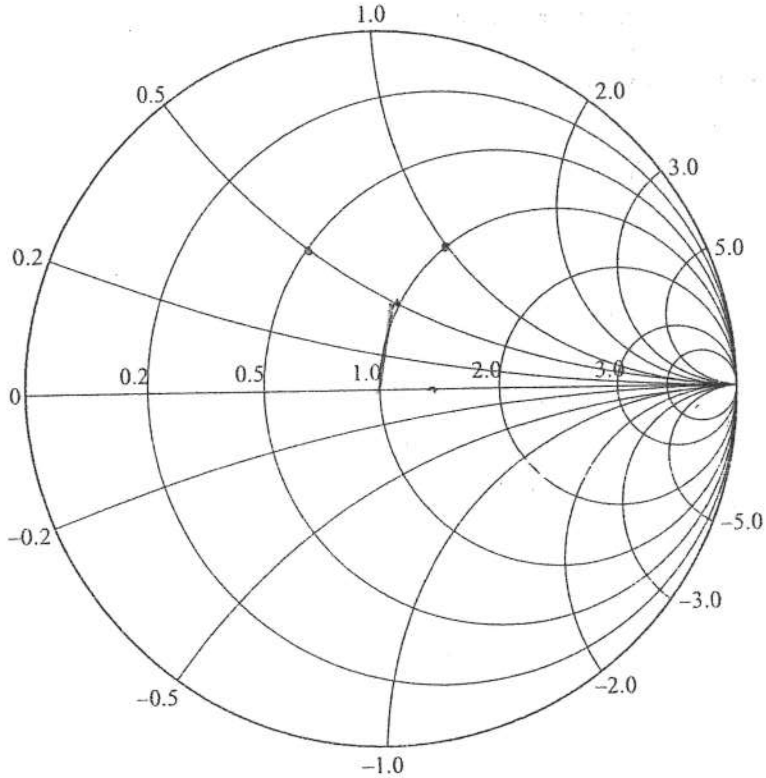


Fig. 10.20 *Smith chart*

10.18 Properties of the Smith Chart

Normalisation of impedance The Smith chart represents the normalised values of R and X circles. If $Z_R = R_R + jX_R$ is the load impedance and $Z_0 = R_0$ is the characteristic impedance of a lossless line, then the R circles give $\frac{R_R}{Z_0}$ values and the X circles give $\frac{X_R}{Z_0}$ values.

Thus, to obtain actual values, the values from the chart should be multiplied by Z_0.

Load impedance plot The intersection points of R and X circles give normalised load impedance (z_R) values. If X is positive, the point is above the $K_R = 0$ axis and if X is negative, the point is below $K_R = 0$ axis.

VSWR plot The VSWR values can be obtained by drawing S-circles on the chart, as shown in Fig. 10.21. The circles with a centre at the origin (0, 0) with radius $|z_R|$ are called S circles.

The radius of the S circle $|z_R|$ gives VSWR value $S = \left|\frac{Z_R}{Z_0}\right|$.

If M is the intersection point of an S circle with the horizontal axis AB, the normalised resistance at M is equal to the value of VSWR. Therefore, $S = \text{OM}$.

$$\frac{Z_{in}}{Z_0} \times \frac{Z_R}{Z_0} = 1 \text{ or } \quad z_{in} \times z_R = 1.$$

$$y_R = z_{in}.$$

Thus, in order to find the admittance on the chart, we first locate the impedance point and rotate it to a distance $\lambda/4$ (quarter wave) towards the generator. The $\lambda/4$ distance is an opposite point on the chart. Hence, the point opposite to the impedance point on the circle gives the admittance point.

Determination of an input impedance: Consider a transmission line of length l terminated with a load impedance Z_R. Denote the normalised load impedance z_R as point P on the Smith chart as shown in Fig. 10.22. With centre O and radius OP, draw the S circle. Extend the OP line to the outer circle which cuts at point P′. Rotate towards the generator (clockwise) up to a length l/λ. Denote this point as N'. Draw the line ON′ which cuts the S circle at point N. This point N represents the normalised input impedance z_{in}.

The angle NOP gives the electric length βl of the line.

Determination of the load impedance: Consider a transmission line of length l. Given the VSWR and the location of the first V_{min} point from the load. To locate the load impedance, first draw the S circle with centre O and radius VSWR. Locate point A on the outer circle at the left side end of the horizontal axis as the position of V_{min}.

Move towards the load (clockwise direction) to a given length l/λ on the wavelength scale and locate the point P. Draw the line OP which cuts the S circle at point P. The location of the point P gives the normalised load impedance.

Input impedance and admittance of an SC (short-circuited) line We know that the input impedance of an SC line is purely reactive and $R = 0$. The short circuit termination represents the position of V_{min}, i.e., a point A on the outer circle at left side end of the horizontal axis (i.e., $R = 0$ on the R circle). From point A, move towards the generator (anti-clockwise direction) to a given length l/λ on the wavelength scale and locate the point P. The location of point P gives normalised input impedance of the SC line. The opposite point Q gives the normalised input admittance.

Input impedance and admittance of an OC (open-circuited) line: We know that the input impedance of an OC line is purely reactive and $R = \infty$. The open circuit termination represents the position of V_{max}, i.e., a point B on the outer circle at a right side end of the horizontal axis (i.e., $R = \infty$ on R-circle). From point B, move towards the generator to a given length l/λ on the wavelength scale and locate the point P. The location of the point P gives normalised input impedance of the OC line. The opposite point Q gives the normalised input admittance.

Determination of locations and lengths of stubs by Smith chart: Design of impedance matching can be easily done by using Smith charts. The locations and lengths of single and double stub matching can be obtained by locating the admittances on the chart. Since the stubs are connected in parallel, it is much easier to combine admittance in parallel than impedances. Also, short-circuited stubs are preferred over open-circuited stubs to avoid radiation losses.

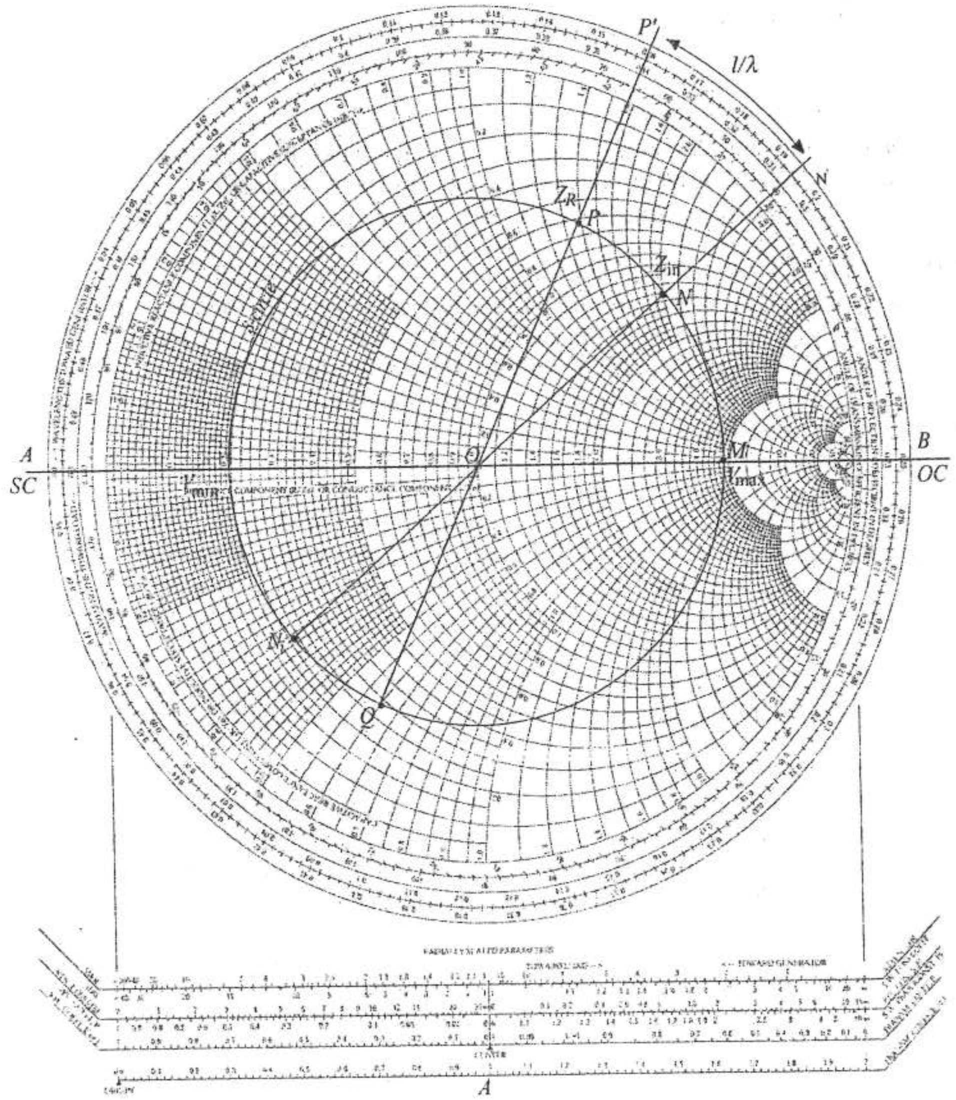


Fig. 10.22 Smith chart for determination of impedances

10.20 Single Stub Matching Using Smith Chart

Consider a transmission line of length l terminated with a load impedance Z_R. Assume that a single stub having a characteristic impedance same as that of the line characteristic impedance and length l_t is connected in parallel to the main line at a distance l_s from the load.

Steps for using Smith chart:

1) Locate the normalised load impedance (z_R) point as P on the chart shown in Fig. 10.23.
2) Locate the normalised load admittance point Q which is opposite to P.

3) Draw the S circle with centre O and radius OP. The length OP on the horizontal axis gives VSWR value before the stub is connected.
4) Locate the point T at the intersection of the S circle with the $R = 1$ circle. There are two intersection points. The point nearer to the load will be taken as the point T.
5) Extend the line OQ on to the outer circle to a point Q′ and extend the line OT on to the outer circle to a point T′. The distance Q′T′ measured on a wavelength scale towards the generator gives the location of the stub on the line from the load.
6) The value of the intersection point T gives the susceptance of the line at the stub location. To nullify the line susceptance, a stub of the same negative susceptance value should be connected. Locating point C with the negative line susceptance on the outer circle gives the susceptance of the stub.
7) The input admittance of the SC stub represents the point B (i.e., $R = \infty$ on the R circle). Measure the distance on the wavelength scale from point B to C. This distance gives the length of the SC stub.
8) Similarly, the input admittance of the OC stub represents the point A (i.e., $R = 0$ on R circle). Measure the distance on the wavelength scale from point A to C. This distance gives the length of the OC stub.

Figure 10.23 shows the Smith chart for determination of length and location of a single stub matching.

Example 10.9 A line characteristic impedance 300 Ω is terminated with a load of $175 + j207$ Ω. An electrical signal of 200 MHz is transmitted along the line in free space.

Determine the following using the Smith chart:

a) Voltage standing wave ratio (VSWR)

b) Load admittance

c) Distance between the load and the first voltage minimum along the transmission line.

Solution Given: Characteristic impedance, $Z_0 = 300\ \Omega$,

Load impedance $Z_R = Z_L = 175 + J207\ \Omega$, frequency $f = 200$ MHz.

$$\therefore \quad \text{Signal wavelength } \lambda = \frac{c}{f} = \frac{3 \times 10^8}{200 \times 10^6} = 1.5 \text{ m.}$$

Steps for using the Smith chart:

1. Normalised load impedance is

$$z_L = \frac{Z_L}{Z_0} = \frac{175 + j207}{300} = 0.58 + j0.69.$$

The point P is located on the Smith chart shown in Fig. 10.24 at the intersection of the circles $R = 0.58$ and $X = 0.69$.

2. Taking the centre as point O and the radius as OP, the impedance circle is drawn. The distance between the centre and point M where the impedance circle crosses the horizontal axis on the right side of the chart gives the voltage standing wave ratio.

VSWR = OM= 2.8.

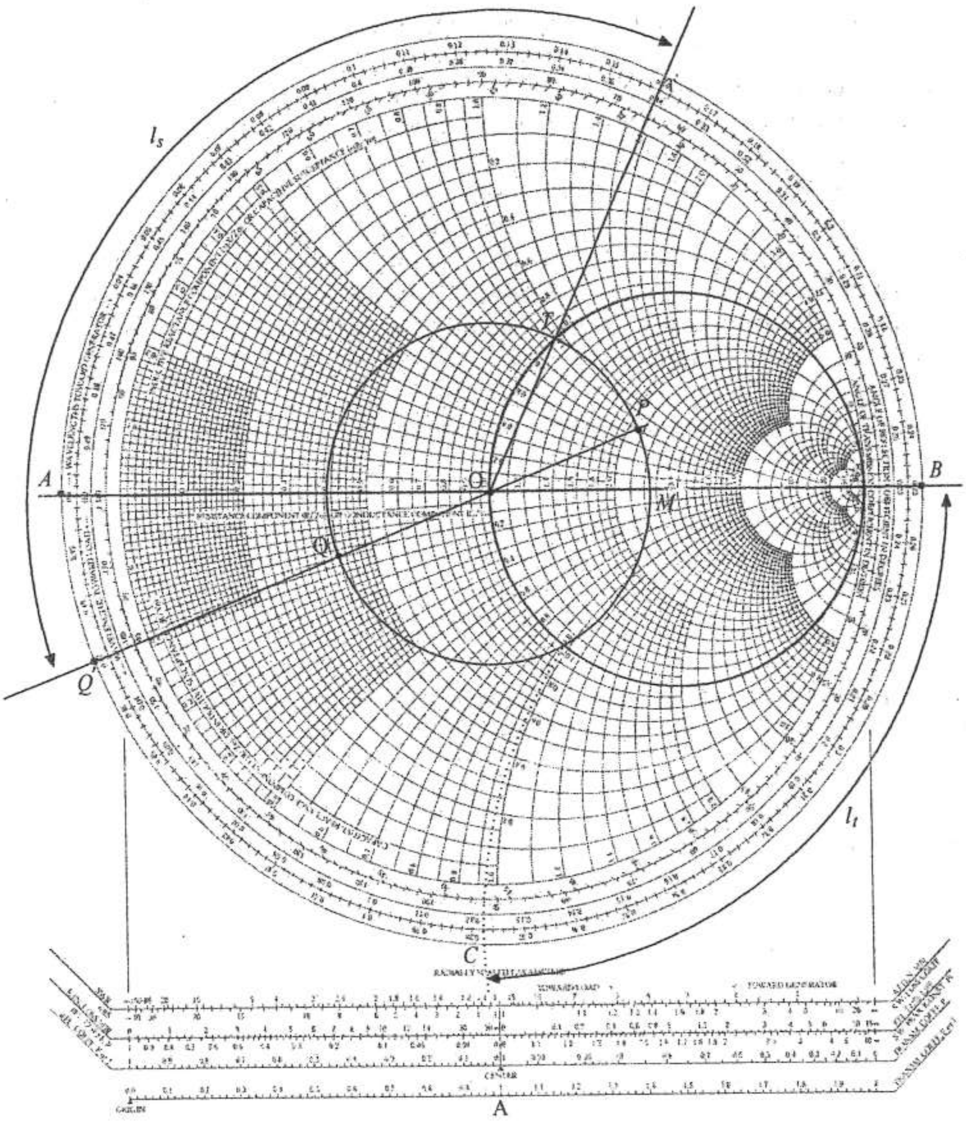


Fig. 10.23 *Smith chart for determination of VSWR, admittance and y_{min}.*

3. The point Q opposite to the normalised impedance point on the impedance circle is the admittance of the load.
   Normalised admittance at point Q is $y_l = 0.72 - j0.85$.
   Load admittance, $Y_L = Y_0 y_l$.

$$Y_L = \frac{1}{300}(0.72 - j0.85) = 2.38 - j2.8 \text{ m℧}.$$

4. The first voltage minimum from the load lies along the horizontal axis to the left side end of the chart, i.e., at point A.

The line OP is extended to the outer circle at point P′. This represents load point.

The position of P′ in terms of the wavelength is 0.114 λ.

The distance between load point P′ and the first voltage minimum point A towards the generator is measured on the wavelength scale. That is,

$$P'A = 0.5\lambda - 0.114\lambda = 0.386\,\lambda.$$

Hence, the distance between the load and the first voltage minimum is

$$y_{\min} = 0.386 \times 1.5 \text{ m.} = 0.58 \text{ m.}$$

Figure 10.24 shows the Smith chart for determining VSWR, admittance and $y_{\min}$.

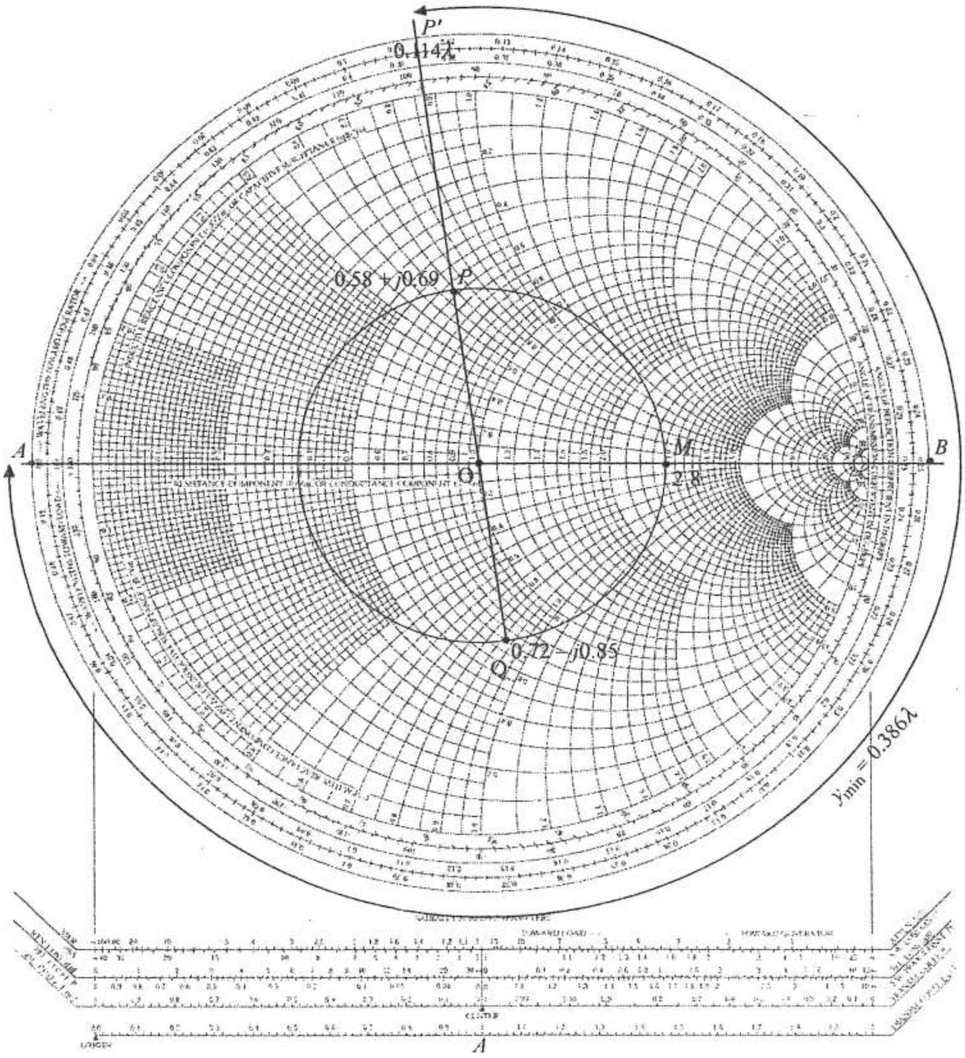


Fig. 10.24 Smith chart for determination of length and location of a single stub.

Example 10.10 Find the location and length of a stub matching for a transmission line having characteristic impedance 50 Ω and terminated with $105 + j40\ \Omega$ using the Smith chart.

Solution Let a short circuited stub of length l_t at a distance l_s from the load be connected parallel to the main line.

Given: Characteristic impedance, $Z_0 = 50\ \Omega$, load impedance, $Z_R = Z_L = 105 + j40\ \Omega$.

Normalised load impedance is

$$z_R = \frac{Z_R}{Z_0} = \frac{105 + j40}{50} = 2.1 + j0.8.$$

Steps for using the Smith chart:

1. The point P is located on the Smith chart shown in Fig. 10.25 at the intersection of the circles $R = 2.1$ and $X = 0.8$.

   The impedance circle is drawn with $O(1 + j0)$ centre and radius OP.

2. The distance between the centre and the point M where the impedance circle crosses the horizontal axis on the right side of the centre gives the voltage standing wave ratio.

   VSWR = length OM = 2.5.

3. Point Q is located diametrically opposite the normalised load impedance point on the impedance circle. It represents normalised load admittance. That is,

   $y_L = 0.415 - j0.158.$

4. Point Q is rotated clockwise to a point T (shorter distance from the load) on the impedance circle where it intersects the $R = 1$ circle. At this point, admittance

   $y = 1 + j0.95.$

5. Line OQ is extended on to the outer circle to a point Q′ and line OT is extended on to the outer circle to point T′. The distance Q′ T′ measured on the wavelength scale towards the generator gives the location of the stub.

   Location of the stub on the line from the load is $l_s = (0.5 - 0.47 + 0.16)\lambda = 0.19\lambda$.

6. The susceptance of the stub is equal to the negative value of the susceptance at point T, i.e., $y_{\text{stub}} = -0.95$. Point C is located on the outer circle for y_{stub}.

7. The length of the short circuited stub is calculated from the right side end of the chart at point B (i.e., $R = \infty$ on the R circle) to point C on the outer circle towards the generator. That is,

   $l_t = 0.379\lambda - 0.25\lambda = 0.13\lambda.$

Figure 10.25 shows the Smith chart for determination of stub matching.

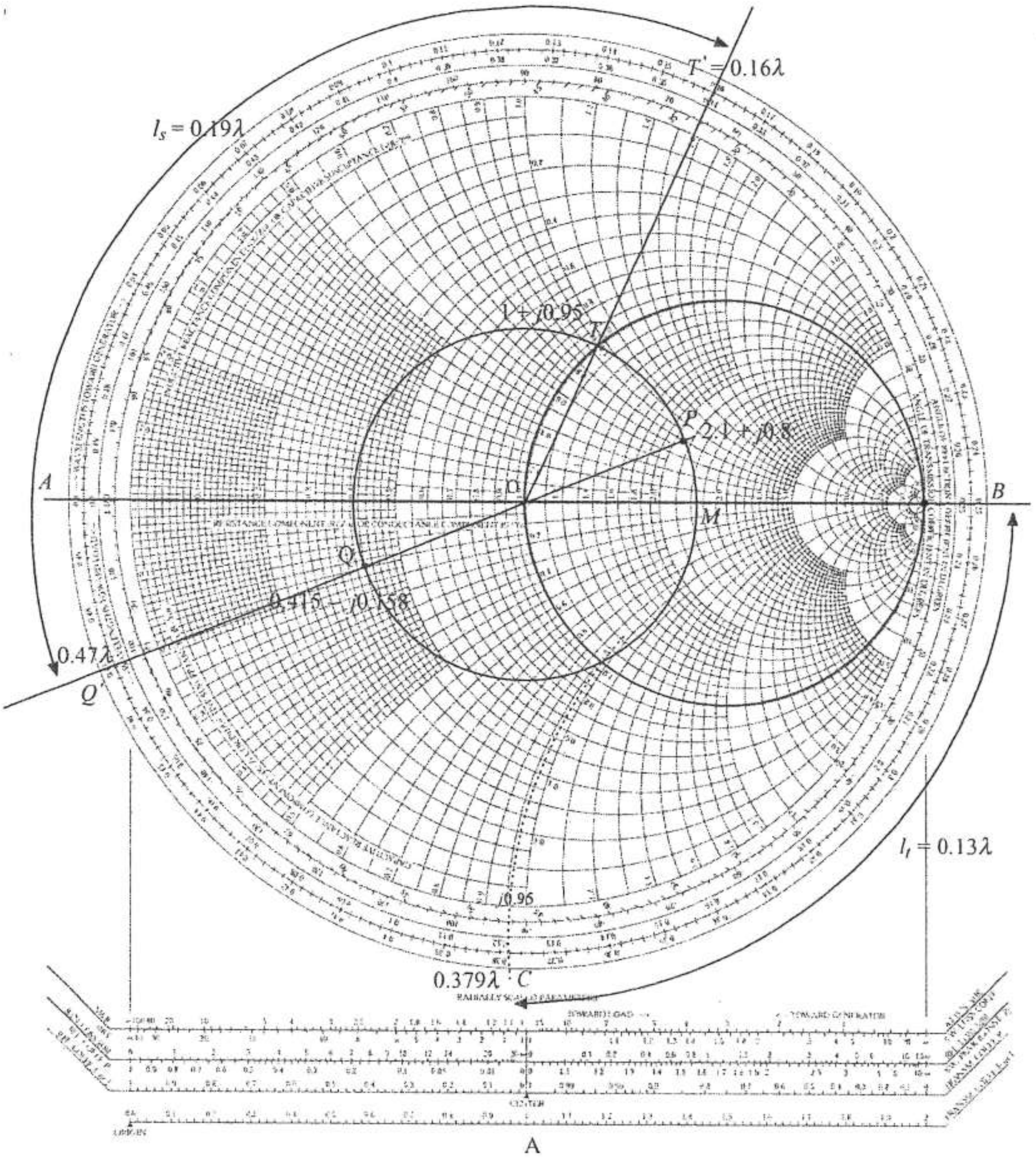


Fig. 10.25 *Smith chart for determination of stub matching*

10.21 Double Stub Matching Using Smith Chart

Consider a transmission line length l terminated with a load impedance Z_R. Assume that two stubs A and B having a characteristic impedance same as that of the line characteristic impedance are connected in parallel nearer to the main line.

Let l_{s_1} and l_{s_2} be the fixed locations of stub A and stub B respectively from the load. Also, let l_{t_1} and l_{t_2} be the lengths of stub A and stub B respectively. Assume that the distance between two stubs is $l_s < \lambda/2$.

The design of a double stub matching is different from a single stub matching. Here the locations of the stubs are fixed as shown in Fig. 10.27. Matching is achieved by adjusting the stub lengths such that the admittance at the second stub is equal to the characteristic admittance. To accomplish this, we use the Smith chart as follows.

Steps for using Smith chart:

1. Locate the normalised load impedance (z_R) point as P.
2. Locate the normalised load admittance point Q which is opposite P.
3. Draw the S circle with centre O and radius OP. The length OP on the horizontal axis gives the VSWR value before the stub is connected.
4. Move the admittance point Q towards the generator (clockwise direction) to a distance l_{s_1} at point Q_1 on the S circle. This point represents normalised admittance at stub A before stub connection.
5. Draw circle 1 for unity conductance ($y = 1$ or $R = 1$ circle) on the right-hand side of the chart for stub B.
6. Since the distance between the two stubs is l_s, all the points of the unity conductance circle should be shifted by l_s distance towards the load. Draw circle 2 to get the shifted unity conductance circle for stub A.
7. Since stub A adds a susceptance to the load, move point Q_1 along the constant conductance circle to a point C on the circle 2. Point C represents the combined normalised admittance at stub A after stub connection.
   Change in susceptance = Susceptance at point C − susceptance at point Q_1.
8. The susceptance of the stub A is equal to the change in susceptance, i.e., y_{stubA}.
   Locate point C′ on the outer circle for y_{stubA}.
9. The length of the short-circuited stub A, l_{t_1}, is calculated from the right side end of chart at point B (i.e., $R = \infty$ on R circle) to the point C′ on the outer circle towards the generator.
10. Draw a circle with centre O and radius OC. It cuts circle 1 at D. Point D gives the admittance on the line at stub B, i.e., y_{s_2} before stub B is connected.
10. To provide proper termination with stub B, the susceptance of the stub B should be equal to the negative value of the susceptance at point D, i.e., y_{stubB}. Locate point D' on the outer circle for y_{stubB}.
12. The length of the short circuited stub B, l_{t_2}, is calculated from the right-side end of the chart at point B (i.e., $R = \infty$ on R circle) to the point D′ on the outer circle towards the generator.

Example 10.11 A lossless transmission line $Z_0 = 300$ ohms has a load impedance of $75 + j225$ ohms. Using a Smith chart, determine what lengths of shorted stubs must be attached to the line to achieve proper matching to maximise the received power.

Solution For proper matching, consider two short circuited stubs connected in parallel to the main line. Let stub A be connected at the load and stub B be connected at a distance $\lambda/4$ from stub B.

Let l_{t_1} and l_{t_2} be the lengths of stub A and stub B respectively.

Movement along the periphery of the chart: On the outer circle or periphery of the chart, moving in the clockwise direction corresponds to travelling from the load towards the generator. Similarly, moving in the anti-clockwise direction corresponds to travelling from the generator towards the load. The full rotation around the chart gives a distance of $\lambda/2$. If the line length is greater than $\lambda/2$, rotate around the circle n times to reach the line length.

Matched load: The circle $R = 1$ represents $z_R = \dfrac{\Re e[Z_R]}{Z_0} = 1$. It passes through the point $(1, 0)$. The resistive part of the load impedance is equal to the characteristic impedance of the line. This circle represents impedance only when the reactive component varies on the line. A stub can be used at this location to nullify the reactive component.

Therefore, the centre point of the chart is known as the matched load point.

10.19 Applications of the Smith Chart

The following are various applications of a Smith chart.

Smith chart as an admittance diagram: Generally, Smith chart is used as an impedance diagram. The impedance Z can be obtained from the intersection of R and X circles. We can also draw the Smith chart for admittance.

Normalised admittance is given by $Y = G + jB$, where $G = \dfrac{1}{R}$ normalised conductance, and $B = \dfrac{1}{X}$ normalised susceptance.

$$\therefore \quad \text{At termination, } y_R = \frac{1}{z_R} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2}.$$

$$\text{So,} \quad G - jB = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2},$$

$$\text{also,} \quad y_R = \frac{1}{z_R} = \frac{1 - K}{1 + K}.$$

$$G - jB = \frac{1 - K}{1 + K}.$$

The G and B circles on the K plane can be drawn similar to R and X circles.

A Smith chart with G and B circles is called an admittance diagram. The admittance diagram is the mirror image of the impedance diagram; all measurements will be taken in the reverse direction.

Converting impedance into admittance: We know that for a lossless quarter wave transformer, if Z_R is the characteristic impedance and Z_0 is the termination impedance, then the input impedance is

$$Z_{in} = \frac{Z_0^2}{Z_R}.$$

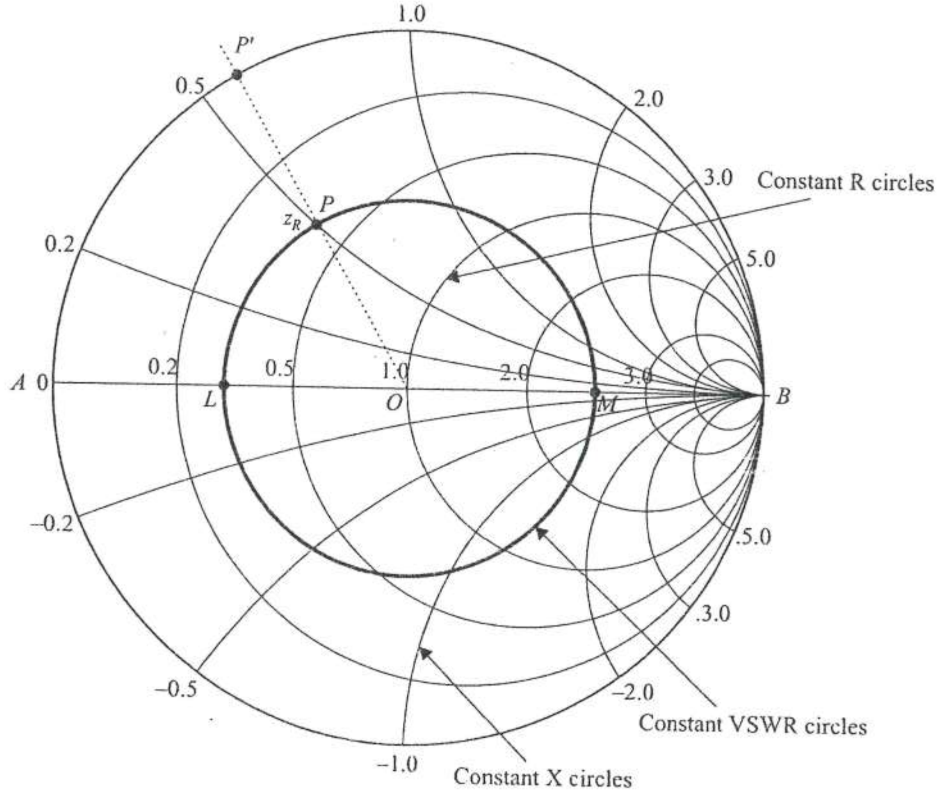


Fig. 10.21 *S circle on Smith chart*

Reflection coefficient values Draw the line OP and extend it to the outer circle, It cuts the outer circle at P$'$. The angles are indicated on the outer circle. The angle (θ) of the line OP gives the angle of the reflection coefficient. The value (K_R, K_X) at the point P gives the magnitude of K. $|K|$ can be obtained from the K scale provided in the chart. The length of OP on the K scale gives the magnitude of K. Also $|K|$ can be calculated directly as $|K| = \dfrac{\text{OP}}{\text{OP}'}$.

Location of voltage maximum and minimum: There are two intersection points of the S circle with the horizontal axis AB. The point at the left side of the centre represents voltage minima $(V_{\min})$ and the point at the right side of the centre represents voltage maxima $(V_{\max})$. Thus the points M and L respectively gives the positions voltage maximum and voltage minimum.

The location of the first $V_{\max}$ can be obtained from the wavelength scale on the outer circle. The arc AP$'$ gives the distance of $V_{\min}$ from the load. Similarly, the arc P$'$AB gives the distance of the first $V_{\max}$ from the load.

Open and short-circuited line: At point B on the right side end of the horizontal axis, both R and X are infinite which represents an open circuit termination of the line. Similarly, at point A on the left side end of the horizontal axis, both R and X are zero which represent short circuit termination of the line.

Problems.

1. A 50Ω Transmission line is terminated by an unknown Impedance. The VSWR is 4 and first minimum is formed at 2cm from the load end. The frequency of operation is 1 GHz. Design a single stub Matching for the above conditions.

Ans : $Z_0 = 50$ VSWR $= 4$ $y_{min} = 2\text{cm}$ $f = 1\text{ GHz}$.

$$K = \frac{S-1}{S+1} = \frac{4-1}{4+1} = \frac{3}{5} = 0.6$$

$$\boxed{2\beta y_{min} - \phi = \pi} \qquad \beta = \frac{2\pi}{\lambda} \qquad \lambda = \frac{c}{f}$$

$$2 \times \frac{20\pi}{3} \times 2 \times 10^{-2} - \phi = \pi \qquad \phi = -2.304 \text{ radians}$$

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = |K| e^{j\phi}$$

$$0.6\, e^{-j2.304} = \frac{Z_R - 50}{Z_R + 50} \qquad Z_R = 25.4\,\Omega$$

$$l_s = \frac{\lambda}{2\pi} \tan^{-1} \sqrt{\frac{Z_R}{Z_0}} = 2.95\text{cm}$$

$$\boxed{l_t = \frac{\lambda}{2\pi} \tan^{-1} \frac{\sqrt{Z_R Z_0}}{Z_R - Z_0}}$$

$$\therefore\ l_t = \underline{4.61 \text{ cm}}.$$

2. A 50 ω lossless line connects a signal of 50 KHz to a load of 140Ω. The load power is 75mW. calculate

i) Voltage Reflection coefficient

ii) VSWR

iii) positions of V_{max}, I_{max}, V_{min}, I_{min}.

Ans : $K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{75-50}{75+50} = 0.2$ $\quad S = \frac{1+0.2}{1-0.2} = 1.5.$

First Maximum voltage amplitude occurs at

$$\boxed{2\beta\, y_{max} - \phi = 0}$$

$2\beta\, y_{max} = 0 \quad \therefore\ y_{max} = 0.$

First voltage Maximum at $y = 0.$

First voltage minimum at $y = \pm \lambda/4.$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{50\times10^3} = 6\ km$$

$y_{min} = 1500m.$

I_{max} occurs at V_{min}

I_{min} occurs at V_{max}

— o —

18. LOSSES IN TRANSMISSION LINES

The losses in transmission lines are of three types.

Losses

Copper losses | Dielectric losses | Radiation losses

I. Copper Losses

These losses occur because of three reasons.

i) I^2R Power Losses :

This is due to dissipation as a result of heating in pure resistance. Its features are:

a) Copper loss is small if there are no current loops. That is, the line should be properly terminated without producing standing waves.

b) Copper loss is more if z_0 is small. This is because, if z_0 is low, current is high. If the current is high, copper loss is more.

ii) Skin Effect :

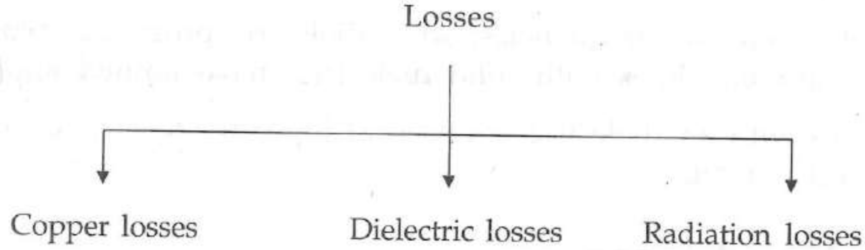
Its features are:

a) When AC signal at high frequency applied to the transmission lines, the current is confined to the surface (skin) of the conductors. This is known as skin effect. This effect reduces cross-sectional area of the conductor with increase in frequency.

b) The decrease in cross-sectional area increases the resistance.

7.18 LOSSES IN TRANSMISSION LINES

Losses in transmission lines are of three types:



Copper Loss

These losses occur because of the following reasons:

1. **I^2R power loss** This is due to dissipation as a result of heating in pure resistance. Its features are:

(a) Copper loss is small if there are no current loops, that is, the line should be properly terminated without producing standing waves.

(b) Copper loss is more if z_0 is small. This is because, if z_0 is low, current is high. If the current is high, copper loss is more.

2. **Skin effect** Its features are:

(a) When an AC signal at high frequency is applied to the transmission lines, the current is confined to the surface (skin) of the conductors. This is known as skin effect. This effect reduces the cross-sectional area of the conductor with increase in frequency.

(b) Decrease in cross-sectional area increases the resistance.

(c) Increased resistance increases power losses.

3. **Crystallisation** Its features are:

(a) Copper losses increase due to ageing of the transmission line. Losses are more when the line is subjected to high temperature, high winds and moisture. Moreover, bending of the line back and forth causes the line to become brittle and cracks appear. This effect is known as crystallisation of the conductors.

(b) Crystallisation increases resistance in the conductors which in turn increases copper losses.

Dielectric Losses

These losses exist due to improper characteristics of dielectric.

Salient features:

(a) These are due to $I^2 R$ power dissipation because of the heating of the solid dielectric material between conductors in transmission lines. These losses are proportional to the voltage across the dielectric.

(b) With increased frequencies, solid dielectric properties worsen and hence transmission lines with solid dielectrics have limited applications.

(c) Lines with air dielectric are used at high frequencies, as air dielectric loss is very small.

Radiation Losses

Salient features:

(a) These losses are high when the spacing between the lines is high as the transmission line acts as an antenna. Therefore, radiation losses are more in parallel-wire lines than in coaxial lines.

(b) At high frequency, λ will be small and hence the transmission lines are not useful at high frequencies.

Electrostatic Fields

UNIT - 1.

Electrostatics is a science related to the electric charges which are static ie. are at rest (or) Electrostatics is the study of time invarient electric fields in a space or vaccum produced by various types of static charge distributions

* Coulomb's law:

Coulomb's law deals with the force a point charge exerts on another point charge. The polarity of charges may be positive or negative; Like charges repel, while unlike charges attract. Coulomb's law states that the force F between two point harges Q_1 and Q_2 is

* Along the line joining them
* Directly proportional to the product Q_1Q_2 of the charges
* Inversely proportional to the square of the distance R betwee them.

Mathematically, $F = \frac{KQ_1Q_2}{R^2}$

K is proportionality constant $= \frac{1}{4\pi\varepsilon}$

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

ε_0 - permittivity in free space

ε_r - permittivity in any medium

For free space or vaccum $\varepsilon_r = 1$

$$\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

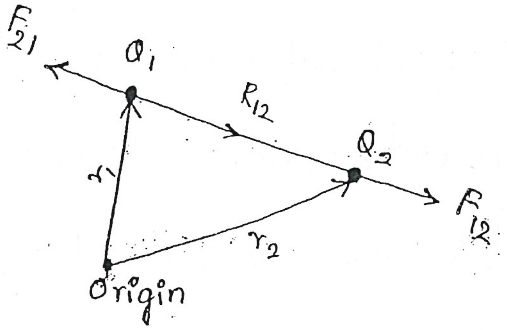
$$k = \frac{1}{4\pi\varepsilon_0} = 9 \times 10^{9} \text{ m/F}$$

$$F = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{R^2}$$

Units for F - Newtons

Q - Coulombs

R - meters



* In vector form:

If point charges Q_1 and Q_2 are located at points having position vectors r_1 and r_2 then the force F_{12} on Q_2 due to Q_1 is

$$F_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} a_{R_{12}} \quad \text{where } R_{12} = r_2 - r_1 \quad R = |R_{12}| \quad a_{R_{12}} = \frac{R_{12}}{R}$$

$$F_{12} = \frac{Q_1 Q_2 \, R_{12}}{4\pi\varepsilon_0 R^2 \, |R|} = \frac{Q_1 Q_2 \,(r_2 - r_1)}{4\pi\varepsilon_0 \, |r_2 - r_1|^3}$$

→ If there are more than two points, we use principle of superposition to determine force on a particular charge. Let $Q_1, Q_2 \cdots Q_N$ be N charges located at points with position vectors $r_1, r_2 \cdots r_N$ the resultant force F is

$$F = \frac{Q}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{Q_k (r - r_k)}{|r - r_k|^3}$$

Consider a small differential length dl carrying a charge dQ, along z-axis hence $dl = dz$

$$dQ = \rho_L \, dl = \rho_L \, dz$$

$$R = (0-0)a_x + (y-0)a_y + (0-z)a_z$$

$$= y a_y - z a_z$$

$$E = \frac{Q}{4\pi\varepsilon_0 R^2} \bar{a}_R$$

$$= \frac{\int_{-\infty}^{\infty} \rho_L \, dz}{4\pi\varepsilon_0 R^2} \cdot \frac{\bar{R}}{|R|} = \frac{\int_{-\infty}^{\infty} \rho_L \, dz}{4\pi\varepsilon_0} \frac{(y a_y - z a_z)}{\left(\sqrt{y^2+z^2}\right)^3}$$

→ For every charge on +ve z-axis there is equal charge present on -ve z axis. Hence z component of electric field intensities produced by such charges cancel each other. Hence effectively there will not be any z component of $\bar{E}$

$$~~dE~~ \quad E = \frac{\int_{-\infty}^{\infty} \rho_L \, dz \, y a_y}{4\pi\varepsilon_0 \left(\sqrt{y^2+z^2}\right)^3}$$

Let $z = y\tan\theta$ $\quad dz = y\sec^2\theta \, d\theta$

$-\infty = y\tan\theta \Rightarrow \theta = -\pi/2$

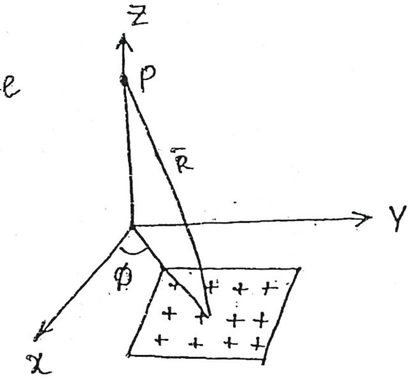
$\infty = y\tan\theta \Rightarrow \theta = \pi/2$

$\therefore$ Limits are $-\pi/2$ to $\pi/2$

$$= \frac{\int_{-\pi/2}^{\pi/2} \rho_L \sec^2\theta \, y \, d\theta}{4\pi\varepsilon_0 \, y^3 \sec^3\theta}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 y} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{\rho_L}{2\pi\varepsilon_0 y} a_y$$

* Electric field intensity due to infinite sheet charge



Consider an infinite sheet of charge having uniform charge density ρ_s placed in x·y plane. The point P at which $\bar{E}$ to be calculated is on z-axis.

$$dQ = \rho_s \, ds$$

$$Q = \int \rho_s \, ds$$

$$E = \frac{Q}{4\pi\varepsilon_0 R^2} \bar{a}_R$$

$$\bar{R} = (0-\rho) a_\rho + (0-0) a_\phi + (z-0) a_z$$

$$= -\rho a_\rho + z a_z$$

$$|R| = \sqrt{\rho^2 + z^2}$$

$$E = \frac{\int \rho_s \, ds}{4\pi\varepsilon_0} \frac{\bar{R}}{R^3} = \frac{\int \rho_s \, ds}{4\pi\varepsilon_0} \frac{(-\rho a_\rho + z a_z)}{\left(\sqrt{\rho^2 + z^2}\right)^3}$$

ρa_ρ term is eliminated

$$E = \frac{\int_0^{\infty}\int_0^{2\pi} \rho_s \, d\rho \, d\phi \cdot \rho \, (z a_z)}{4\pi\varepsilon_0 \left(\sqrt{\rho^2 + z^2}\right)^3}$$

Consider $\rho^2 + z^2 = u^2$

$2\rho \, d\rho = 2u du$

Limits are z to ∞ and 0 to 2π

$$E = \frac{\int_z^{\infty}\int_0^{2\pi} \rho_s \, \cancel{u} du \, d\phi \, (z a_z)}{4\pi\varepsilon_0 \; u^{3\,2}}$$

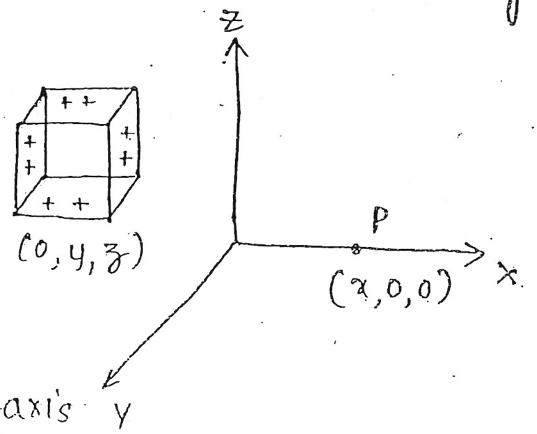
$$= \frac{\rho_s}{4\pi\varepsilon_0} \int_z^{\infty} \frac{1}{u^2} du \int_0^{2\pi} d\phi \; z a_z$$

$$= \frac{\rho_s}{4\pi\varepsilon_0} \left(-\frac{1}{u}\right)_z^{\infty} (\phi)_0^{2\pi} \; z a_z$$

$$= \frac{\rho_s}{2\,4\pi\varepsilon_0} \, 2\pi \times a_z = \boxed{\frac{\rho_s a_z}{2\varepsilon_0}}$$

* Electric field intensity due to infinite volume charge

Consider infinite volume charge cube in YZ plane and the point P where electric field intensity is to be found is on X-axis



$$\rho_v = \frac{dQ}{dv}$$

$$dQ = \rho_v \, dv$$

$$Q = \int \rho_v \, dv$$

$$\bar{r} = x a_x - y a_y - z a_z$$

$$|r| = \sqrt{x^2+y^2+z^2}$$

$$E = \frac{\int \rho_v \, dv \; \bar{r}}{4\pi\varepsilon_0 R^3} = \frac{\int \rho_v \, dv}{4\pi\varepsilon_0} \; \frac{x a_x - y a_y - z a_z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

Eliminating y and z components

$$E = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{\rho_v \, \cancel{r^2} \sin\theta \, dr \, d\theta \, d\phi}{4\pi\varepsilon_0 \left(\sqrt{x^2+y^2+z^2}\right)^{\cancel{3}}} \; x a_x \qquad \left(\because \sqrt{x^2+y^2+z^2} = r\right)$$

$$= \frac{\rho_v}{4\pi\varepsilon_0} \int_0^{\pi} \sin\theta \, d\theta \cdot \int_0^{r} \frac{1}{r} \, dr \cdot \int_0^{2\pi} d\phi \; x a_x$$

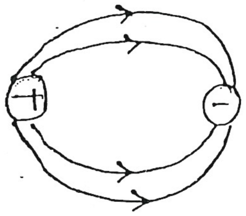
$$= \frac{\rho_v}{4\pi\varepsilon_0} \left(-\cos\theta\right)_0^{\pi} \left(\ln r\right)_0^{r} \left(\phi\right)_0^{2\pi} \; x a_x$$

$$= \frac{\rho_v}{4\cancel{\pi}\varepsilon_0} \times \cancel{2} \times (\ln r - 1) \times 2\cancel{\pi}$$

$$= \boxed{\frac{\rho_v}{\varepsilon_0} (\ln r - 1) \, x a_x}$$

Electric flux: Electric flux is the total number of lines of force in any particular field. It is denoted by ψ. It's units are coulomb (C).

→ The flux lines start from positive charge and terminate on negative charge. The flux lines are parallel and never cross each other



* Electric flux density: The number of flux lines passing through unit surface area is called electric flux density. It is is denoted by D. Units are C/m^2

$$D = \frac{\text{Flux}}{\text{Unit surface}} = \frac{d\psi}{ds}$$

→ This is also called displacement flux density or displacement density.

* Gauss's law: It constitutes one of the fundamental laws of electromagnetism. It states that the total electric flux ψ through any closed surface is equal to the total charge enclosed by that surface.

i.e $\boxed{\psi = Q}$

* It satisfies only when all charges are symmetrical

* Relation between Electric flux density (D) and electric field intensity (E) :=

We know that $D = \frac{d\psi}{ds}$

$$d\psi = D\,ds$$

$$\psi = \int D\,ds$$

From gauss law $\psi = Q \Rightarrow Q = \int D \cdot ds$

$$Q = D\int ds \qquad (\because ds = r^2 \sin\theta\, d\theta\, d\phi)$$

$$Q = D \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta\, d\theta\, d\phi$$

$$= Dr^2 \int_0^{\pi} \sin\theta\, d\theta \cdot \int_0^{2\pi} d\phi$$

$$Q = 4\pi D r^2$$

$$\therefore D = \frac{Q}{4\pi r^2} \qquad \left(\text{since } E = \frac{Q}{4\pi\varepsilon_0 r^2}\right)$$

$$\Rightarrow \boxed{D = \varepsilon_0 E}$$

* Proof of Gauss law:

We know that $D = \frac{d\psi}{ds}$

$$\psi = \int D \cdot ds$$

$$\psi = \int \frac{Q}{4\pi r^2} \, ds \qquad \left(\because D = \frac{Q}{4\pi r^2} \right)$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{Q}{4\pi r^2} r^2 \sin\theta \, d\theta \, d\phi$$

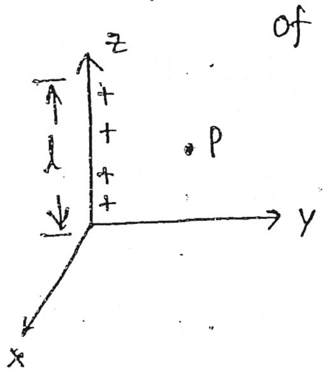
$$= \frac{Q}{4\pi} \cdot \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta$$

$$= \frac{Q}{4\pi} \times 2\pi \times 2$$

$$\therefore \boxed{\psi = Q}$$

* Electric flux density due to infinite line charge (Application of Gauss l

Consider infinite line charge along z-axis. We find electric flux density at point P



We know that $D = \frac{d\psi}{ds}$

$$\psi = \int D \cdot ds$$

According to Gauss law $\psi = Q$

$$Q = \int D \cdot ds$$

$$\int \rho_L \, dl = \int D \cdot ds$$

$$\int \rho_L \cdot dz = \int D \cdot ds$$

$$\rho_L \cdot l = D \oint ds$$

$$\rho_L \cdot l = D \int_0^{2\pi} \int_{-\infty}^{\infty} d\phi \, dz \, \rho$$

$$= D\rho \int_0^{2\pi} d\phi \cdot \int_{-\infty}^{\infty} dz$$

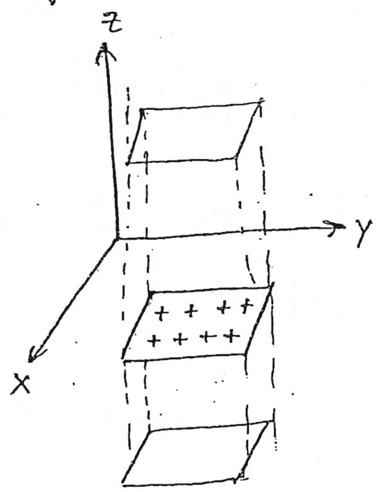
$$\rho_L \cdot l = D\rho \times 2\pi \times l$$

$$D = \frac{\rho_L}{2\pi\rho}$$

$$\boxed{\bar{D} = \frac{\rho_L}{2\pi\rho} a_\rho}$$

* Electric flux density due to sheet charge:

Consider the plate in XY plane



$$D = \frac{d\psi}{ds}$$

$$\psi = \int D \, ds$$

$$\int \rho_s \, ds = \int D \, ds$$

$$\iint \rho_s \cdot dx \, dy = \int D \, ds$$

$$\rho_s \cdot A = \int_{top} D \, ds + \int_{bottom} D \, ds$$

$$\rho_s \cdot \cancel{A} = 2D \int \cancel{ds}$$

$$\boxed{D = \frac{\rho_s}{2}}$$

* Electric flux density due to volume charge:

We know that $D = \frac{d\psi}{ds}$

$$\psi = \int D \cdot ds$$

$$Q = \int D \cdot ds$$

$$\int \rho_v \, dv = \int D \cdot ds$$

$$\rho_v \cdot \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi = \int D \cdot ds$$

$$\rho_v \cdot \int_0^{r} r^2 \cdot \int_0^{\pi} \sin\theta \, d\theta \cdot \int_0^{2\pi} d\phi = D \int ds.$$

$$\rho_v \cdot \frac{r^3}{3} \times 2 \times 2\pi = D \int_0^{\infty}\int_0^{\pi} r \, dr \, d\theta$$

$$= D \cdot \int_0^{a} r \, dr \cdot \int_0^{\pi} d\theta$$

$$\frac{4\pi \rho_v r^3}{3} = D \cdot \frac{a^2}{2} \times \pi.$$

$$\boxed{\therefore D = \frac{8 \rho_v r^3}{3a^2}}$$

* Derivation of Maxwell equation using Gauss law:

According to Gauss law

$$\psi = Q$$

and we know that $\psi = \int D \cdot ds$

$$Q = \int D \cdot ds$$

$$\int \rho_v \, dv = \int D \cdot ds \qquad \left(\because Q = \int \rho_v \, dv \text{ from volume charge density} \right)$$

According to Divergence theorem

$$\int A \cdot ds = \int \nabla \cdot A \, dv$$

$$\therefore \int D \cdot ds = \int \nabla \cdot D \, dv$$

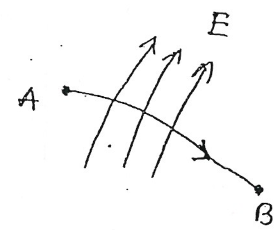
$$\therefore \int \rho_v \, dv = \int \nabla \cdot D \, dv$$

$$\Rightarrow \boxed{\rho_v = \nabla \cdot D}$$ → 1^{st} maxwell equation in differential for

In integral form $\psi = \int D \cdot ds$

* Electric potential: It is defined as the work done per unit charge in moving unit charge from one point to another point. It is denoted by V. Units are

Joule/Coulomb or Volts



$$V = \frac{W}{q}$$

Suppose we wish to move a point charge Q from A to B in an electric field E.

UNIT- 1.

∇ is defined in Cartesian coordinates

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

1. The gradient of a scalar V is ∇V
2. The divergence of a Vector A is $\Delta \cdot A$
3. The curl of a vector is $\nabla \times A$
4. Leplaucin of a scalar V is $\nabla^2 V$.

Gradient of a Scalar :- The gradient of a Scalar field V is a vector that represents both magnitude and direction of Max space rate of increase of V

$$\text{grad } V = \nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

$$\boxed{F = -\nabla V}$$

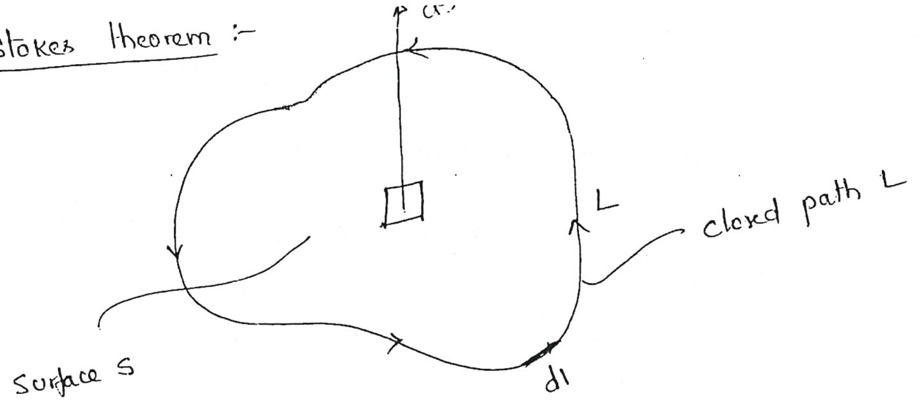
curl :- $\nabla \times V$ denotes how much the vector V curls around the point in Question.

The curl of A is an axial vector whose magnitude is the Maximum circulation of A per unit area tends to zero whose direction is the normal direction of area when the area is oriented so as to make circulation Maximum.

$$\nabla \times A = \text{curl } A = \left(\lim_{\Delta s \to 0} \frac{\oint_L A \cdot dL}{\Delta s} \right) a_{n\,max}$$

where the area Δs is bounded by the curve L and a_n is the unit vector normal to the surface Δs and it is determined by using right hand rule

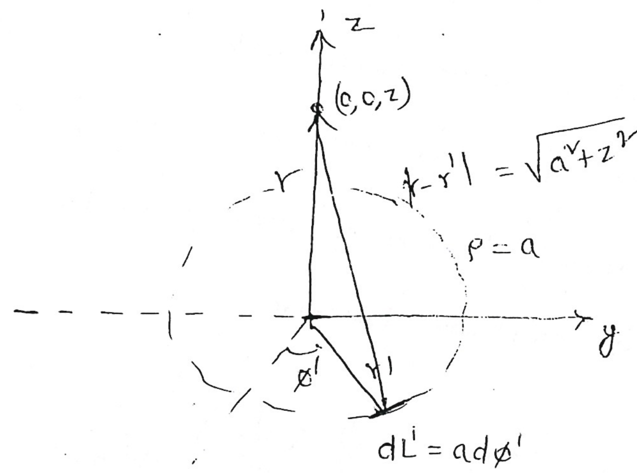
Stokes theorem :-



$$\oint_L A \cdot dl = \int_S (\nabla \times A) \cdot ds$$

statement : It states that the circulation of a vector field A around a closed path L is equal to the surface integral of the curl of A over the open surface S bounded by L provided that A and $\nabla \times A$ are continuous on S.

Potential Field of a ring of uniform line charge density.



$$V = \int \frac{\rho_L(r')\,dL'}{4\pi\epsilon_0 |r-r'|} = \int_0^{2\pi} \frac{\rho_L\, a\, d\phi'}{4\pi\epsilon_0 \sqrt{a^2+z^2}} = \frac{\rho_L a}{2\epsilon_0 \sqrt{a^2+z^2}}$$

Relation between E and V :- $\boxed{E = -\nabla V}$

where ∇V = gradient of potential.

$$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z \quad \text{(cartesian)}$$

$$\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z \quad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} a_\phi \quad \text{(spherical)}$$

Problems : If $V = 2x^2y - 5z$ Find V at $P(-4, 3, 6)$ and E and D and ρ_v.

$$V_p = 2(-4)^2(3) - 5\times6 = 66\ V$$

$$E = -\nabla V$$

$$= -\left[4xy\,a_x + 2x^2 a_y - 5a_z\right] = -4xy\,a_x - 2x^2 a_y + 5a_z \ \ V/m$$

$$E_p = 48\,a_x - 32a_y + 5a_z \ \ V/m$$

$$|E_p| = \sqrt{(48)^2 + (-32)^2 + 5^2} = 57.9\ V/m$$

Direction of E at P is $a_{E,p} = \dfrac{48a_x - 32a_y + 5a_z}{57.9}$

$$= 0.829\,a_x - 0.553a_y + 0.086a_z$$

$$D = \epsilon_0 E$$

$$= -35.4\,xy\,a_x - 17.71\,x^2 a_y + 44.3a_z \ \ pC/m^2$$

$$\nabla\cdot D = \rho_v$$

$$\rho_v = -35.4y \ \ pC/m^3$$

Potential : Potential at a point is the potential difference between the point and a reference point at which the potential is zero.

For point charges $$V(r) = \frac{1}{4\pi\epsilon_0} \sum_{K=1}^{n} \frac{Q_K}{|r - r_K|}$$

For line charge $$V(r) = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_L(r')\, dl'}{|r - r'|}$$

For surface charge $$V(r) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_S(r')\, ds'}{|r - r'|}$$

For Volume charge $$V(r) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho_v(r')\, dv'}{|r - r'|}$$

1. If the reference point is not infinity then

$$V = \frac{Q}{4\pi\epsilon_0 r} + c$$

where c is const to be determined at the reference point.

2. If the charge distribution is known above formulas are used

If E is known then we have

$$V = -\int E \cdot dL + c$$

Problems :- Two point charges are located at $(2, -3, 3)$ and $(0, 4, -2)$ respectively. The charges are $-4\mu C$, $5\mu C$. Find the potential at $(1, 0, 1)$ assuming zero potential at infinity.

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} + C_0$$

If $V(\infty) = 0$ $C_0 = 0$

$$|r - r_1| = \sqrt{6} \qquad |r - r_2| = \sqrt{26}$$

$$V(1, 0, 1) = \frac{10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}} \right] = -5.872 \text{ kV}$$

Problem: If $V = \frac{60 \sin\theta}{r^2}$ in free space and the point is $(r = 3m, \theta = 60^\circ, \phi = 25^\circ)$

find a) V_P b) E_P c) $\left.\frac{dv}{dN}\right|_P = \text{grad } V$ d) a_N at P e) ρ_v at P

Solution: a) $V_P = \frac{60 \sin 60^\circ}{9} = 5.77 V$

b) $E = -\nabla V = -\left[\frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} a_\phi\right]$

$= -\left[60\sin\theta \cdot \frac{-1}{r^3} a_r + \frac{1}{r} \cdot \frac{60}{r^2}\cos\theta + 0\right]$

$E_P = 3.85 a_r - 1.111 a_\theta \ V/m$.

c) $\nabla V = 3.85 a_r - 1.111 a_\theta \ V/m$

$|\nabla V| = \sqrt{(3.85)^2 + (1.111)^2} = 4.01 \ V/m$

d) a_N at P $\frac{-(3.85 a_r - 1.111 a_\phi)}{4.01} = -0.961 a_r + 0.277 a_\phi$

e) $\nabla \cdot D = \rho_v$ $D = \epsilon_0 E$

$D = \epsilon_0 [3.85 a_r - 1.111 a_\phi]$

$D = 34.08 a_r - 9.8377 a_\phi \ pC/m^2$

$\nabla \cdot D = \rho_v = -7.57 \ pC/m^3$

3) Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$

Find D at $(2, \pi/2, 0)$

$$E = -\nabla V$$

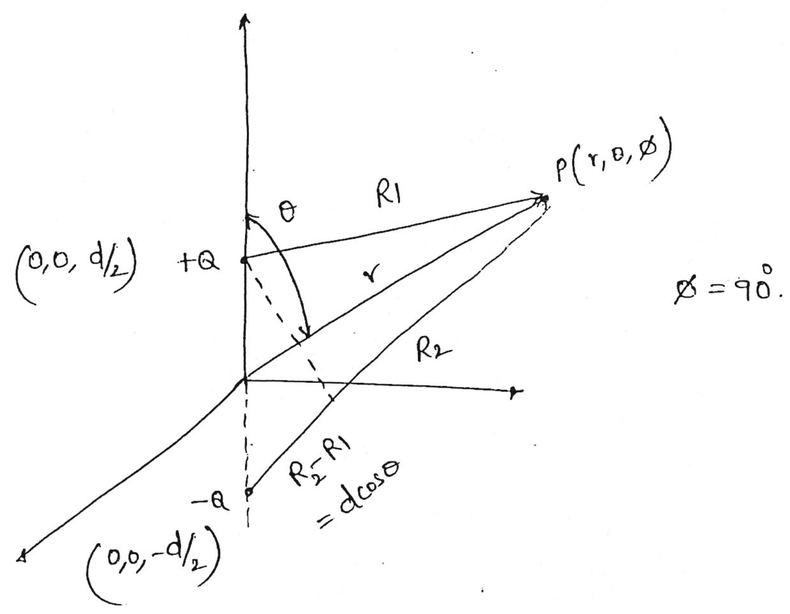
$$= \frac{20}{r^3} \sin\theta \cos\phi \, a_r - \frac{10}{r^3} \cos\theta \cos\phi \, a_\theta + \frac{10}{r^3} \sin\phi \, a_\phi$$

$$D = \epsilon_0 E(r=2, \theta=\pi/2, \phi=0)$$

$$= \epsilon_0 \left[\frac{20}{8} a_r - 0 a_\theta + 0 a_\phi \right]$$

$$= 2.5 \epsilon_0 a_r \, C/m^2 = 22.1 a_r \, pC/m^2$$

Dipole :



It is the basis for behaviour of Dielectric materials.

Dipole : Similar (Two) charges of equal Magnitude and opposite sign sepperated by small distance d.

$$V_p = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2} \right] = \frac{Q}{4\pi\epsilon_0} \frac{R_2 - R_1}{R_1 R_2} = \frac{Q}{4\pi\epsilon_0} \frac{d\cos\theta}{r^2}$$

for $z=0$ plane and $\theta = 90^\circ$ $V = 0$.

$$E = -\nabla V$$

$$\therefore E = -\left[\frac{Qd\cos\theta}{2\pi\epsilon_0 r^3} a_r - \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3} a_\theta\right]$$

$$= \frac{Qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\, a_r + \sin\theta\, a_\theta\right]$$

Dipole moment $\boxed{\bar{p} = Q\bar{d}}$

Since $d \cdot a_r = d\cos\theta$

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{Q\, d\cdot a_r}{4\pi\epsilon_0 r^2} = \frac{p\cdot a_r}{4\pi\epsilon_0 r^2}$$

$$\therefore \boxed{V = \frac{1}{4\pi\epsilon_0 |r-r'|^2}\, p \cdot \frac{r-r'}{|r-r'|}}$$

Point dipole is achieved when $d \to 0$ and $Q \to \infty$

So that p is finite.

$V\,(\text{dipole}) \propto \frac{1}{r^2}$ $\quad V\,(\text{Quadropole}) \propto \frac{1}{r^3}$ $\quad V\,(\text{octupole}) \propto \frac{1}{r^4}$

$E\,(\text{dipole}) \propto \frac{1}{r^3}$ $\quad E\,(\text{Quadropole}) \propto \frac{1}{r^4}$

$E\,(\text{octupole}) \propto \frac{1}{r^5}$

~~Problems~~.

Electric Flux line : It is an imaginary path or line drawn in such a way that its direction at any point is the direction of Electric field at that point.

Problems : Two dipoles with dipole moments $-5a_z$ nc/m and $9a_z$ nc/m are located at point $(0,0,-2)$ and $(0,0,3)$ respectively. Find the potential at the origin.

$$V = \sum_{K=1}^{2} \frac{P_K \cdot r_K}{4\pi\epsilon_0 |r-r_K|^3}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{P_1\cdot r_1}{r_1^3} + \frac{P_2 \cdot r_2}{r_2^3}\right]$$

$P_1 = -5a_z$ $\quad$ $r_1 = (0,0,0) - (0,0,-2) = 2a_z$ $\quad$ $|r_1| = 2$

$P_2 = 9a_z$ $\quad$ $r_2 = (0,0,0) - (0,0,3) = -3a_z$ $\quad$ $|r_2| = ?$

$$V = \frac{1}{4\pi \cdot \frac{10^{-9}}{36\pi}}\left[\frac{-10}{2^3} - \frac{27}{3^3}\right]10^{-9}$$

$$= -20.25\,V$$

Energy density in Electrostatic Fields :-

$$W_E = \frac{1}{2}\sum_{m=1}^{N} Q_m V_m$$

where $V_1, V_2, \ldots$ are the total potentials at those points.

For continuous charge distribution

$$W_E = \frac{1}{2}\int_{vol} \rho_v V\, dv$$

$$W_E = \frac{1}{2}\int_{vol} D\cdot E\, dv = \frac{1}{2}\int_{vol} \epsilon_0 E^2\, dv$$

$$\left[\because \nabla\cdot VA = A\cdot\nabla V + V(\nabla\cdot A)\right]$$

$$dW_E = \frac{1}{2} D\cdot E\, dv$$

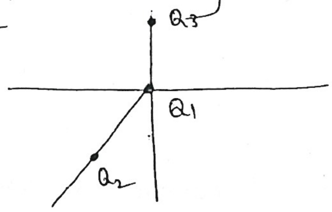
$$\frac{dW_E}{dv} = \frac{1}{2} D\cdot E = \text{energy density } (\text{Joules}/m^3)$$

<u>Problem</u>: Three point charges $-1nc$, $4nc$, $3nc$ are located at $(0,0,0)$ $(0,0,1)$, and $(1,0,0)$. Find energy in the system.

$$W = W_1 + W_2 + W_3$$

$$W = \frac{1}{2}\sum_{K=1}^{3} Q_K V_K = \frac{1}{2}\left[Q_1V_1 + Q_2V_2 + Q_3V_3\right)$$

$$= \frac{Q_1}{2}\left[\frac{Q_2}{4\pi\epsilon_0(1)} + \frac{Q_3}{4\pi\epsilon_0(1)}\right] + \frac{Q_2}{2}\left[\frac{Q_1}{4\pi\epsilon_0 1} + \frac{Q_3}{4\pi\epsilon_0\sqrt{2}}\right]$$



$$+ \frac{Q_3}{2}\left[\frac{Q_1}{4\pi\epsilon_0(1)} + \frac{Q_2}{4\pi\epsilon_0\sqrt{2}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right)$$

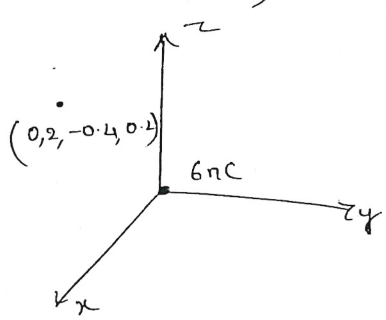
$$= 13.37\,nJ.$$

<u>Problem on potential</u> :-

A point charge of $6nc$ is located at the origin. Find potential at $P(0.2, -0.4, 0.2)$ given that a) $V=0$ at ∞
b) $V=0$ at $(1,0,0)$

$$C = 0.$$

$$V_P = \frac{6\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}\sqrt{(0.2)^2+(0.4)^2+(0.2)^2}} = 110V$$



b) $c' \neq 0$

$$0 = \frac{Q}{4\pi\epsilon R} + C$$

$$C = \frac{-Q}{4\pi\epsilon R} = \frac{-6\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}\sqrt{1^2+0^2+0^2}} = -54V$$

$$V_P = \frac{Q}{4\pi\epsilon R} - 54$$

$$= 110 - 54 = \underline{56V}.$$

Electric Fields in Material Science :- up to now the fields are discussed in free space. If we extend this concept to Materials all the old equations are alsomost valid with minor modifications. Now we are going to study the fields in conductors and Dielectrics.

Convection current :- The current through a given area is the electric charge passing through area per unit time

$$I = \frac{dQ}{dt}$$

A current is said to be 1 Ampere when 1 coloumb of charge is passing through Reference point per second.

If ΔI current passes through the surface ΔS then

$$J_n = \frac{\Delta I}{\Delta S}$$

$$\therefore \quad \Delta I = J_n \Delta S$$

where J_n is the current density $\perp$r to the surface. If it is not normal to surface then

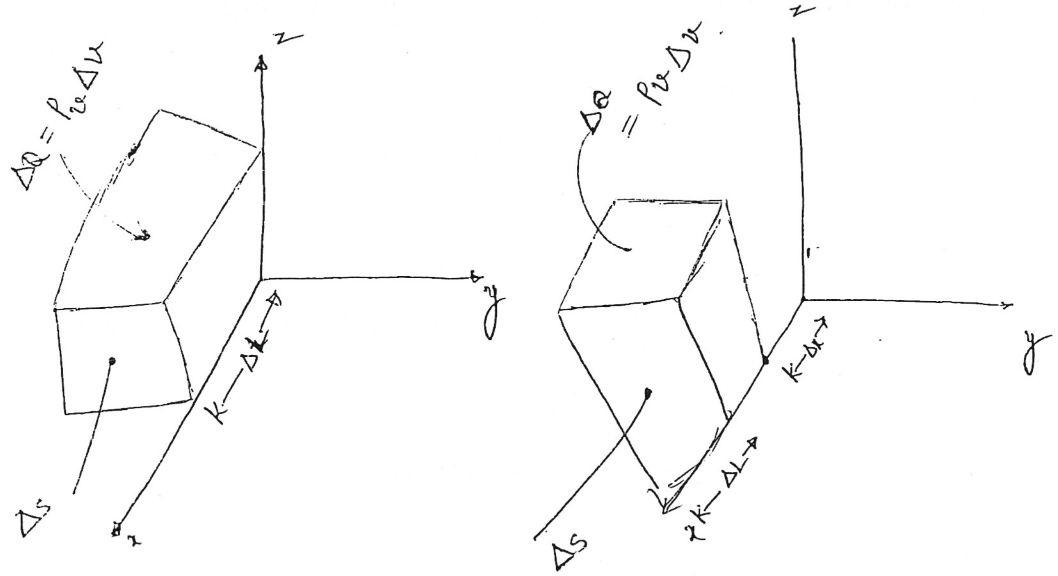
$$\Delta I = J \cdot \Delta S$$

$$\boxed{I = \int_S J \cdot ds}$$

Depends on how I is produced, there are 3 current densities.

1. Convection current density
2. Conduction current density
3. Displacement current density.

Convection current density :- It does not involve conductors and does not satisfy ohms law. A beam of electrons in vaccum tube is called convection current, which is the best example.



Consider the element of charge

$$\Delta Q = \rho_v \Delta v = \rho_v \cdot \Delta S \, \Delta L$$

of the block moves Δx in time Δt

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \cdot \frac{\Delta x}{\Delta t}$$

$$\therefore \quad \Delta I = \rho_v \, \Delta S \cdot V_x$$

$$\frac{\Delta I}{\Delta S} = J_x = \rho_v \cdot V_x$$

where $\rho_v \cdot V_x$ is called convection current density

Problem :- If $J = \frac{4}{r^2}\cos\theta\, a_r + 20 e^{-2r}\sin\theta\, a_\theta - r\sin\theta\cos\phi\, a_\phi$ A/m^2

1. Find J at $r=3, \theta=0, \phi=\pi$

$$J = \frac{4}{9} a_r = 0.444\, a_r \; A/m^2$$

2. Find the total current through the spherical cap $r=3$, $0<\theta<20^\circ$

$0<\phi<2\pi$ in the a_r direction.

$$I = \int_S J \cdot ds$$

$$= \int_S \left(\frac{4}{r^2}\cos\theta\, a_r + 20e^{-2r}\sin\theta\, a_\theta - r\sin\theta\cos\phi\, a_\phi\right) \cdot r^2 \sin\theta\, d\theta\, d\phi\, a_r$$

$$= \int_S \frac{4}{\cancel{r^2}}\cos\theta \cdot \cancel{r^2}\sin\theta\, d\theta\, d\phi$$

$$= 4\int_0^{2\pi}\int_0^{20}\cos\theta\sin\theta\, d\theta\, d\phi$$

$$= 1.470 \text{ Amps.}$$

Continuity of current :- outward flow of +ve charge must

balanced by decrease of +ve charge within the closed surface.

$$I = \oint_S J \cdot ds = -\frac{dQ_i}{dt}$$

$$\oint J \cdot ds = \int_{vol} \nabla \cdot J\, dv = -\frac{d}{dt}\int_{vol} \rho_v\, dv$$

$$\nabla \cdot J\, dv = -\frac{\partial \rho_v}{\partial t} dv$$

$$\boxed{\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}}$$

For static fields $\boxed{\nabla \cdot J = 0}$

Relaxation time is the time it takes a charge placed in the interior of material to drop to $e^{-1} = 36.8\%$ of its initial value.

Conduction current density : It requires a conductor. A conductor is characterised by large amount of free electrons the provide conduction current due to an impressed electric field.

$$\therefore \quad F = -eE$$

$$\boxed{J = \sigma E} \rightarrow \text{point form of ohms law.}$$

Properties of conductor :-

1. A perfect conductor can not contain an electrostatic field within it.

A conductor is a equipotential body. $E = -\nabla V = 0$

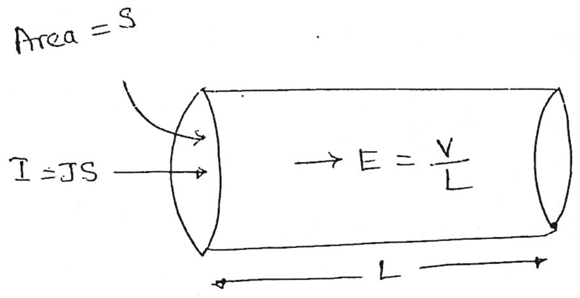
From ohms law $J = \sigma E$

For finite J we have to take $\sigma \rightarrow \infty$ and E should vanish inside the conductor.

i.e $E \rightarrow 0$ because $\sigma \rightarrow \infty$ in a perfect conductor

Under static conditions

$$\boxed{E = 0 \quad \rho_v = 0 \quad V_{ab} = 0}$$ inside a conductor.



The above ckt is a conductor which is maintained at a potential difference V. In this case $E \neq 0$ inside a conductor. There is no static equilibrium because it is connected to external EMF. So Electric field must exist to sustain the flow of current.

Resistance of conducting material :- The direction of E is same as the flow of +ve charges.

$$E = \frac{V}{L}$$

If the cross sectional area is S then $J = \frac{I}{S}$

$$\therefore \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$$

$$\therefore \frac{V}{I} = R = \frac{L}{\sigma S}$$

$$\boxed{R = \frac{\rho . L}{S}}$$ where ρ = resistivity of material

For non uniform cross section

$$R = \frac{V}{I} = \frac{\oint E \cdot dL}{\int \sigma E \cdot ds}$$

$$P = \int E \cdot J \, dv \quad = \text{Joules law}$$

$$w_p = \frac{dp}{dv} = E \cdot J = \sigma |E|^2 \text{ watts}/m^3.$$

For conductor with uniform cross section

$$dv = ds \, dL$$

$$P = \int_L E dl \int_S J ds = VI$$

$$\boxed{P = I^2 R}$$ $\longrightarrow$ Joules law in ckt theory.

Problem :- If $J = \frac{1}{r^3}\left(2\cos\theta\, a_r + \sin\theta\, a_\theta\right)$ A/m^2 calculate the current passing through

a) A hemi spherical cell of radius 20cm

b) A spherical shell of radius 10cm .

Sol : a) $I = \int J \cdot ds$ $\qquad ds = r^2 \sin\theta\, d\theta\, d\phi\; a_r$ in this case

$$I = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \frac{1}{r^3} 2\cos\theta\; r^2 \sin\theta\, d\theta\, d\phi \Big|_{r=0.2}$$

$$= \frac{2}{r} \cdot 2\pi \int_{\theta=0}^{\pi/2} \sin\theta\, d(\sin\theta) \Big|_{r=0.2} = 31.4A.$$

b) The only difference is take $\theta = 0$ to $\theta = \pi$ and $r = 0.1$

$\therefore$ $I = 0$

Problem : For the current density $J = 10z \sin^2\phi \, a_\rho$ A/m^2. Find the current through the cylindrical surface $\rho = 2$ $\quad 1 \le z \le 5$ m

Ans . 754A .

Find the current in the circular wire shown in the figure If the current density is $J = 15\left(1 - e^{-1000r}\right) a_z$ A/m^2. The radius of wire is 2mm . A

$dI = J \cdot ds$

$= 15\left(1 - e^{-1000r}\right) a_z \cdot \rho\, d\rho\, d\phi\, a_z$

$$I = \int_0^{2\pi} \int_0^{0.002} 15\left(1 - e^{-1000r}\right) \rho\, d\rho\, d\phi$$

$= 0.133$ mA .

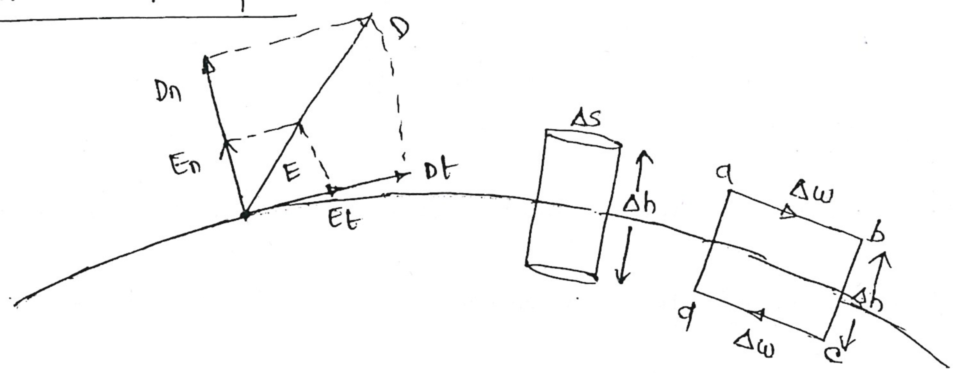
Boundary conditions :- So we have considered the fields in homogenious medium. Of the field exists in a region consisting of two different media the conditions that the field must satisfy at the interface are called boundary conditions. They are helpful to determine the field on one side of boundary when other side field is given. There are three interfaces to be discussed.

1) conductor to free space

2) conductor to Dielectric

3) Dielectric to Dielectric

Conductor to free space :-



$(E=0)$

$$\oint E \cdot dL = 0$$

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

$E_t \Delta w = 0 \qquad E_t = 0 \qquad D_t = 0$

For Normal components

$$\oint_S D \cdot ds = Q$$

$$\int_{top} + \int_{Bottom} + \int_{sides} = Q$$

$$D_N = \rho_s$$

For conductor — free space condition

$$\boxed{\begin{array}{l} D_t = E_t = 0 \\ D_N = \epsilon_0 E_N = \rho_s \end{array}}$$

As $E_t = 0$ it is equipotential surface.

<u>Summary</u> : a) Electric field intensity inside a conductor is zero.

b) E at the surface of a conductor is everywhere directed normal to the surface.

c) The conductor surface is equipotential surface.

<u>Problem</u> :- If V is given that $V = 100(x^2 - y^2)$ and the point $P(2,-1,3)$ is on conductor, free space boundary

$$\therefore V_p = 100(4-1) = 300\,V$$

Therefore 300V exists ~~on~~ ~~sur~~ every where

The locus of points where a potential of 300V exits is

$$300 = 100(x^2 - y^2)$$

$$x^2 - y^2 = 3$$ This is the equation of the conductor surface.

$$\therefore E = -\nabla V = -100\,\nabla(x^2 - y^2)$$

$$= -200x\,a_x + 200y\,a_y$$

$$\therefore E_p = -400\,a_x - 200\,a_y \ \ V/m$$

$$D = \epsilon_0 E$$

$$D_p = -3.54\,a_x - 1.77\,a_y \ \ nC/m^2$$

$$D_N = |D_p| = 3.96\ nC/m^2$$

Dielectric Materials.

Bound charges :- The dielectric is an Electric field can be viewed as free space arrangement of Microscopic electric dipoles. These are not free charges and they can not contribute to the conduction. The bound charges are other sources of Electric field. The energy will be stored by shifting the relative positions of Dielectric materials.

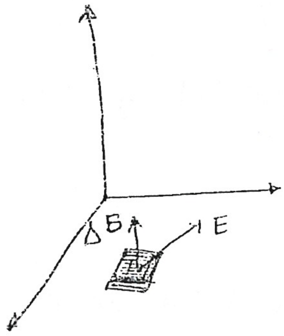
If there are n dipoles per unit volume and we deal with a volume Δv and there are $n\Delta v$ dipoles and the total dipole moment is

$$P_{total} = \sum_{i=1}^{n\Delta v} P_i$$

Polarisation · It is the dipole movement per unit volume

$$P = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} P_i$$

units · are columbs/ sqmetre.



$$Q_b = -\oint_S P \cdot ds$$

$$Q_T = \oint_S \epsilon_0 E \cdot ds$$

$$Q_T = Q_b + Q$$

where Q = total free charge enclosed by the surface.

$$\therefore Q = Q_T - Q_b = \oint (\epsilon E + P) \cdot ds$$

$$D = \epsilon_0 E + P$$

$$Q = \oint_S D \cdot ds$$

For volume charge densities we have

$$Q_b = \int_v \rho_b \, dv$$

$$Q = \int_v \rho_v \, dv$$

with the help of divergence theorem we have

$$\nabla \cdot P = -\rho_b$$

$$\nabla \cdot \epsilon_0 E = \rho_T$$

$$\boxed{\nabla \cdot D = \rho_v}$$

Relation between E and P :-

Isotropic Material is one in which E and P are linearly related. In this materials E and P are always parallel, regardless of the orientation of field.

Ferro ~~magne~~electric Material : In this E and P are non linear and shows Hysteresis effects. Polarisation produced by a given electric field depends on past history of the Sample.

Ex : Barium titanate, Rochell salt.

$$\boxed{P = \chi_e \epsilon_0 E}$$

where χ_e is electric susceptability and dimensionless.

$$\therefore D = \epsilon_0 E + P$$

$$= \epsilon_0 E + \chi_e \epsilon_0 E$$

$$= (\chi_e + 1)\, \epsilon_0 E$$

$$\therefore \epsilon_R = (\chi_e + 1)$$

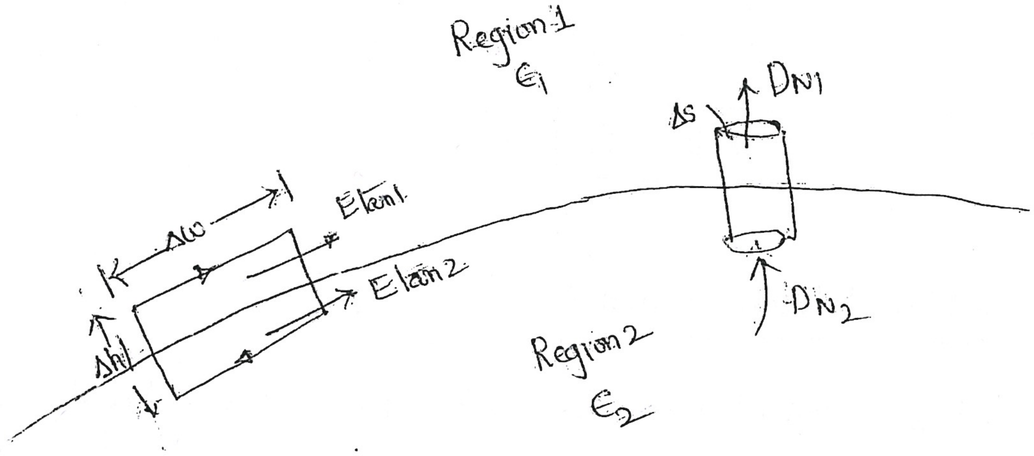
$$\therefore D = \epsilon_0 \epsilon_R E$$

$$\boxed{D = \epsilon E}$$

$$\boxed{\begin{array}{l} \epsilon = \epsilon_0 \epsilon_r \\ \epsilon_r = 1 + \chi_e = \dfrac{\epsilon}{\epsilon_0} \end{array}}$$

where ϵ = permittivity.

Boundary conditions For perfect Dielectric Materials :-



For tangential components $\oint E \cdot dL = 0$

$$E_{tan1} \Delta w - E_{tan2} \Delta w = 0 .$$

$$\therefore \boxed{E_{tan1} = E_{tan2}} \longrightarrow \text{continuous}$$

$$\boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\epsilon_1}{\epsilon_2}} \longrightarrow \text{dis continuous} .$$

For Normal components : $D_{N1} \Delta S - D_{N_2} \Delta S = \Delta Q = \rho_s \Delta S$

$$\boxed{D_{N_1} - D_{N_2} = \rho_s}$$

For ρ_s = Zero on the Interface

$$D_{N_1} = D_{N_2} \longrightarrow \text{continous} ,$$

$$\boxed{\frac{E_{N_1}}{E_{N_2}} = \frac{\epsilon_2}{\epsilon_1}} \longrightarrow \text{dis contin}$$

Boundary conditions between conductor and Dielectric :- replace ϵ_0 by ϵ

$$\therefore \quad \boxed{\begin{array}{l} D_t = E_t = 0 \\ D_N = \epsilon E_N = \rho_s \end{array}}$$

poisson's and Leplace's equations :-

we know that $\nabla \cdot D = \rho_v$

$D = \epsilon E$

$E = -\nabla V$

$\therefore \nabla \cdot D = \nabla \cdot \epsilon E = -\nabla \cdot \epsilon \nabla V = \rho_v$

$\nabla \cdot \nabla V = \frac{-\rho_v}{\epsilon}$ for a homogenious region where ϵ is constant.

The above equation is poisson's equation

$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ $\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$

$\therefore \nabla \cdot \nabla V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

$$\boxed{\therefore \nabla \cdot \nabla V = \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho_v}{\epsilon}}$$

If we make $\rho_v = 0$ then

$\nabla^2 V = 0$ which is Leplace's equation

∇^2 operation is called Leplacian of V

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{cartesian}$$

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \quad \text{(cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2}$$

(Spherical)

The dielectric constant (or relative permittivity) ϵ_r is the ratio of the permittivity of the dielectric to that of free space.

The dielectric strength is the maximum electric field that a dielectric can tolerate or withstand without breakdown.

Problem: A certain homogenious slab of lossless dielectric material characterized by an electric susceptibility of 0.12 and carries a uniform D within it of $1.6\, nC/m^2$ Find

a) E

$$D = \epsilon_0 \epsilon_R (E)$$

$$D = \epsilon_0 (\chi_e + 1) E$$

$$E = \frac{D}{\epsilon_0 (\chi_e + 1)} = \frac{1.6 \times 10^{-9}}{\frac{1}{36\pi} \times 10^{-9} (0.12 + 1)} = 161.3\ V/m.$$

b) P

$$P = \chi_e \epsilon_0 E$$

$$= 0.12 \times 8.854 \times 10^{-12} \times 161.3 = 171.4\ PC/m^2$$

c) average dipole moment if there are 2×10^{19} dipoles per cubic meter.

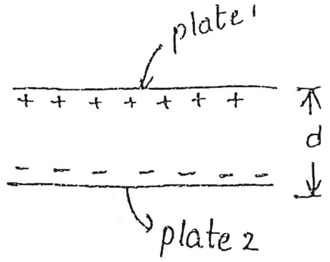
$$\text{Average dipole moment} = \frac{P}{\text{avera}}$$

$$= \frac{171.4 \times 10^{-12}}{2 \times 10^{19}} = 8.57 \times 10^{-30}\ C.$$

* Different types of capacitors:

1. Parallel plate capacitor:

It consists of two parallel plates seperated by a distance d. The space between plates is filled with dielectric of permittivity ε



Let A be the area of cross-section

$$C = \frac{Q}{V}$$

$$Q = \int \rho_s \, ds = \rho_s \int ds = \rho_s \cdot A$$

$$V = \int_1^2 E \, dl = E \int_1^2 dl = E \cdot d$$

$$\therefore \quad C = \frac{\rho_s A}{E d} \quad — ①$$

From Gauss law $D = \frac{d\psi}{ds} \Rightarrow \psi = \int D \, ds$

$$\Rightarrow Q = \int D \cdot ds \quad (\because \psi = Q)$$

$$\therefore \int \rho_s \, ds = \int D \cdot ds$$

$$\therefore D = \rho_s$$

$$\varepsilon E = \rho_s$$

Substitute above eqn in ①

$$\Rightarrow \quad C = \frac{\varepsilon \cancel{E} \cdot A}{\cancel{E} d} \quad \Rightarrow \quad \boxed{C = \frac{\varepsilon A}{d}}$$

Energy stored in a capacitor:

$$C = \frac{dQ}{dV}$$

$$V = \int E\, dl$$

$$V = E \cdot d \implies E = V/d$$

$$W = 1/2 \int \epsilon E^2 = 1/2 \int \epsilon \frac{V^2}{d^2}\, dv \qquad (\because dv = ds \cdot dl)$$

$$= 1/2 \int \epsilon \frac{V^2}{d^2}\, ds\, dl$$

$$= 1/2\ \epsilon \frac{V^2}{d^2} \oint ds\, dl$$

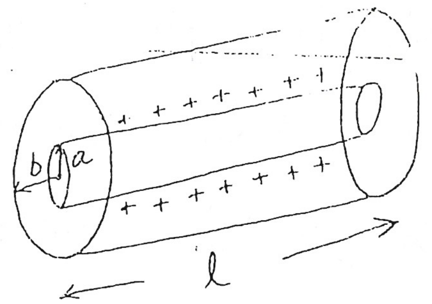
$$= 1/2\ \epsilon \cdot \frac{V^2}{d^{\cancel{2}}} \times A \times \cancel{d}$$

$$= 1/2\ \frac{\epsilon A}{d} V^2$$

$$= 1/2\ C V^2$$

2. Coaxial cable type of capacitor:

Consider length l of two coaxial conductors of inner radius a and outer radius b. The space between the conductors is filled with homogenous dielectric with permittiv ε.



We know that $C = \frac{Q}{V}$

$$Q = \int \rho_L \, dl = \rho_L \cdot l$$

$$V = \oint E \, dl$$

$$= \int \frac{\rho_L}{2\pi\varepsilon_0 y} \cdot dy$$

$\left(\because \text{ Electric field intensity due to infinit line charge} = \frac{\rho_L}{2\pi\varepsilon_0 y}\right)$

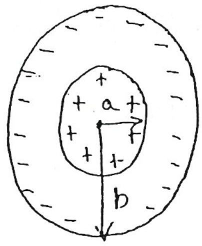
$$= \frac{\rho_L}{2\pi\varepsilon_0} \left(\ln y \right)_a^b$$

$$= \frac{\rho_L}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{\cancel{\rho_L} \cdot l}{\frac{\cancel{\rho_L}}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right)}$$

$$\boxed{C = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{b}{a}\right)}}$$

3 Spherical capacitor: A spherical capacitor is the case of two concentric spherical conductors. Consider inner sphere of radius a and outer sphere of radius b (b>a), seperated by dielectric medium with permittivity ε



$$C = \frac{Q}{V}$$

$$V = \int E\, dl$$

$$V = \int \frac{Q}{4\pi\varepsilon_0 r^2}\, a_r \cdot dr$$

$$= \frac{Q}{4\pi\varepsilon_0} \int_a^b \frac{1}{r^2}\, dr = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)_a^b = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$\boxed{C = \frac{4\pi\varepsilon_0\, ab}{b-a}}$$

→ For an isolated capacitor i.e $b \to \infty$ $\therefore$ $\boxed{C = 4\pi\varepsilon_0 a}$

Magnetic Field.

There are two major laws governing in Magnetostatic fields. They are

1) Biot-Savart Law : It is like columb's law and it is law of magnetostatics

2) Ampere's circuital Law : It is special case of Biot-Savart law and it is applied for Symmetrical current contribution. It is like a Gauss law in Electrostatics.

(Ampere's law for a current element)

Biot-Savart law :- It states that at any point P the magnitude of Magnetic field intensity produced by the differential element is proportional to the product of current, the magnitude of the differential length, and the sine of angle lying between the filament and line connecting the filament to the point P where the field is desired. The Magnitude of H is inversely proportional to square of the distance from differential element to 'P'. The direction of H is normal to the plane containing differential element and the line drawn from filament to 'P'.

The Normal is to be chosen which is the direction of progress of right handed screw turned from dL through smaller angle to the line from the element to 'P'.

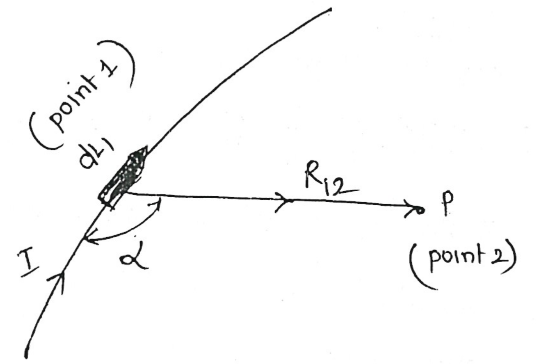
$$dH \propto \frac{I dl \sin\alpha}{R^2}$$

$$dH = \frac{K I dl \sin\alpha}{R^2} \qquad K = \frac{1}{4\pi}$$

$$\therefore dH = \frac{I\, dL \times a_R}{4\pi R^2}$$

$$= \frac{I dL \times a_R}{4\pi R^2} = \frac{I dL \times R}{4\pi R^3}$$

Unit of H is (A/m) $\qquad H = \oint \frac{I dL \times a_R}{4\pi R^2}$

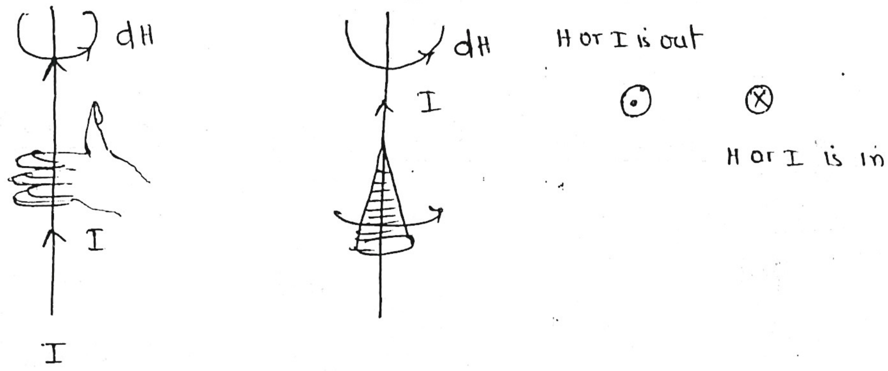


$$dH_2 = \frac{I_1 \, dL_1 \times a_{R12}}{4\pi R_{12}^2}$$

Biot savart law is also called as <u>Ampere's law for the current element.</u>

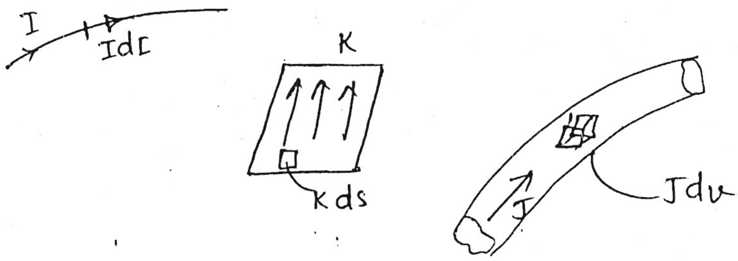
Magnetic field is due to both time varying current and DC current. But for the present discussion we consider DC current only.

we can not varify Biot - Savart law because the differential current element can not be isolated. The direction of dH can be determined by the right hand role with Right hand thumb pointing in the direction of current, the right hand fingers encircling the wire in the direction of dH. with the screw should be placed along the wire and pointed in the direction of current flow, the direction of advanced screw is in the direction of dH.



we have 3 types of different current distributions

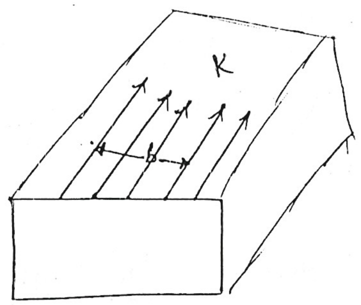
1. Line current
2. Surface current
3. Volume current.



Biot - Savart Law in terms of Distributed Sources :-

Surface current density (K) :- It is current which flows in a sheet of Vanishingly small thickness and measured in Ampers / metre width. If the surface current density is uniform $I = Kb$

where b = width $\perp$r to the direction in which current is flowing.



For non uniform ~~charge~~ surface current density

$$I = \int K\,dN$$

where dN = differential element of path across which current flows.

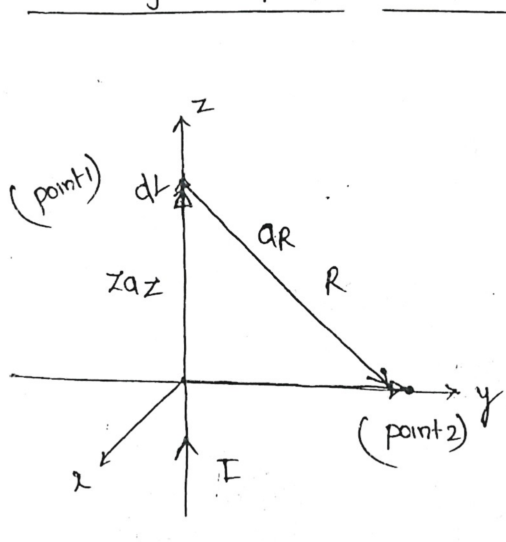
$$\boxed{I\,dL = K\,ds = J\,dv}$$

$$H = \int_L \frac{I\,dL \times a_R}{4\pi R^2} \quad \text{(Line current)}$$

$$\therefore \quad H = \int_S \frac{K \times a_R\, ds}{4\pi R^2} \quad \text{(Surface current)}$$

$$H = \int_{vol} \frac{J \times a_R\, dv}{4\pi R^2} \quad \text{(Volume current)}$$

The Magnetic field Intensity due to long straight filament :-



No variation of H with z or ϕ.

we choose a point in $z = 0$ plane.

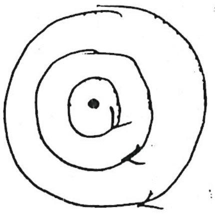
$$H_2 = \int_{-\infty}^{\infty} \frac{I\,dz'\, a_z \times (\rho a_\rho - z' a_z)}{4\pi\,(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\, dz'\, a_\phi}{(\rho^2 + z'^2)^{3/2}}$$

$$\boxed{H_2 = \frac{I}{2\pi\rho} a_\phi}$$

The Magnitude of H varies inversely as the distance from filament.

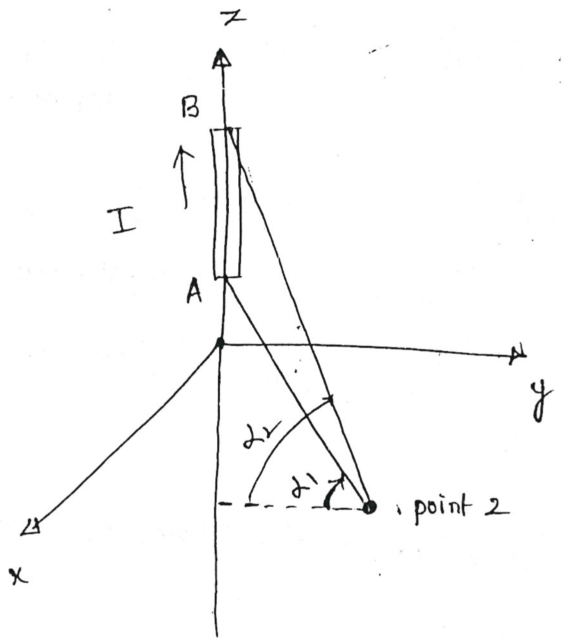
The stream lines are circles about the filament. The seperation of stream lines proportional to radius and inversely proportional to H.



I = into the page.

≈ equipotential of electric field.

Magnetic field Intensity due to finite length current filament on Z axis :—



$$H = \frac{I}{4\pi\rho}\left(\sin\alpha_2 - \sin\alpha_1\right) a_\phi$$

If one or both points are below point α_1 or both α_1 and α_2 are negative.

Different types of Magnetic materials :

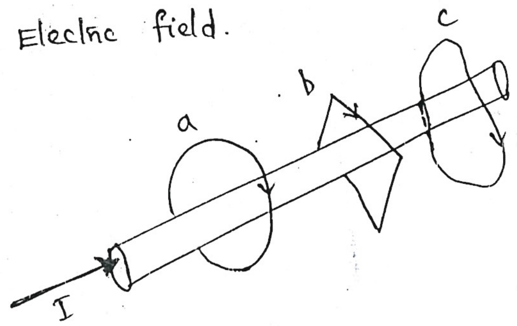
1) diamagnetic → Bismuth.
2) paramagnetic → increase in B, potassium, oxygen, tungsten.
3) Ferromagnetic → large dipole moment
4) anti ferro magnetic → atomic moments will be in antiparallel fashion
5) ferri magnetic → Same ↑ but moments are not equal.
6) super paramagnetic → assemblage of ferro magnetic particles.

(Ampere's work law):-

<u>Ampere's circuital Law</u> :- It is also called as Ampere's work Law.

$$\boxed{\oint H \cdot dL = I}$$

It states that the line integral of H about any closed path is exactly equal to the direct current enclosed by that path. +ve current is in the direction of right handed screw turned in a direction in which closed path is traversed. It is equivalent to Gauss law in Electric field.

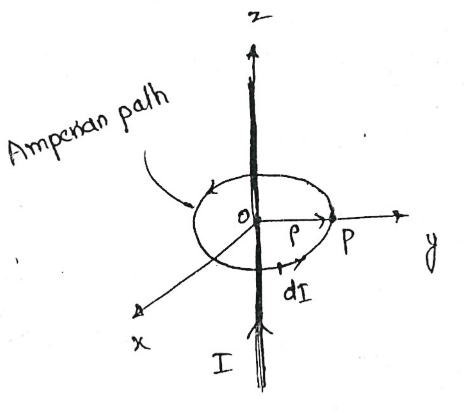


For the path c

$\oint H \cdot dL < I$.

The above law is applicable when current distribution is symmetrical.

<u>Applications of Ampere's Law</u> :- 1. Infinite Line current.



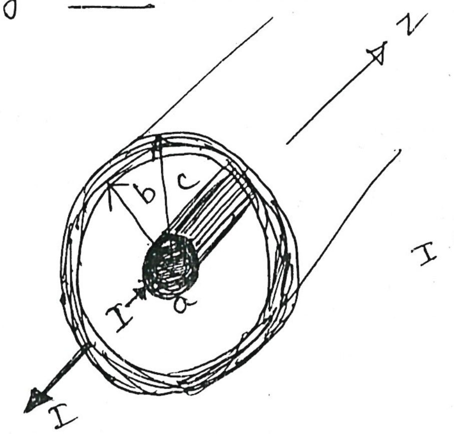
consider an infinitely long filament current along z axis. H is const provided ρ is constant. Since this path encloses whole current I, According to Ampere's Law

$$I = \oint H \cdot dL = \int H_\phi a_\phi \cdot \rho \, d\phi \, a_\phi$$

$$H_\phi \int \rho\, d\phi = H_\phi \cdot 2\pi\rho = I$$

$$\therefore \boxed{H_\phi = \frac{I}{2\pi\rho}}$$

2. Infinitely long coaxial Transmission Line :-



I enters in Inner conductor

Leaves in outer conductor.

In the above case H_ϕ component Varies with ρ

A circular path of radius ρ, where ρ is larger than radius of inner conductor but less than inner radius of outer conductor

$$H_\phi = \frac{I}{2\pi\rho} \quad (a<\rho<b)$$

If $\rho < a$ then

$$I_{enclosed} = I \cdot \frac{\rho^2}{a^2}$$

$$2\pi\cancel{\rho}\, H_\phi = I \cdot \frac{\rho^{\cancel{2}}}{a^2}$$

$$H_\phi = \frac{I\rho}{2\pi a^2} \quad (\rho<a)$$

$$H_\phi = 0 \quad (\rho>c)$$

$$H_\phi = \frac{I}{2\pi\rho} \cdot \frac{c^2-\rho^2}{c^2-b^2} \quad (b<\rho<c)$$

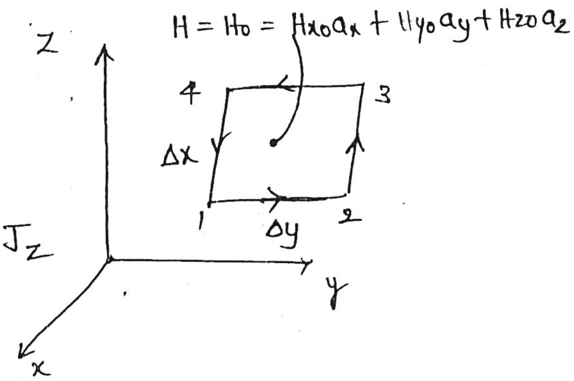
Point form of Ampere's circuital Law :- Ampere's work law in Differential vector Form.

Definition of curl :-

$$(\text{curl } H)_N = \lim_{\Delta S_N \to 0} \frac{\oint H \cdot dL}{\Delta S_N}$$

where ΔS_N is the planar area enclosed by the closed line integral.

N indicates that it is the component which is Normal to the surface enclosed by closed path.



Statement :-

$$\lim_{\Delta x \Delta y \to 0} \frac{\oint H \cdot dL}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

The curl of any vector is a vector, and any component of curl is given by the limit of the quotient of closed line integral of a vector about a small path in a plane normal to that component desired and area enclosed as the path shrinks to zero.

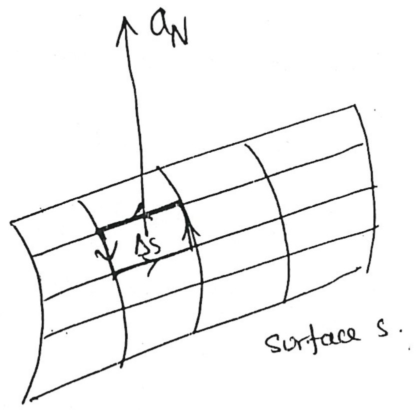
$$\therefore \text{curl } H = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \nabla \times H.$$

point form of Ampere's circuital Law is $\boxed{\nabla \times H = J}$

For conservative fields $\oint E \cdot dL = 0$

$$\boxed{\nabla \times E = 0}$$

Stokes Theorem :-



we Know $\dfrac{\oint H \cdot dL}{\Delta S} \doteq (\nabla \times H)_N$

Stokes theorem relates surface integral to the closed line integral.

$$\oint H \cdot dL \equiv \int_S (\nabla \times H) \cdot ds$$

$$\oint H \cdot dL = I = \oint_S J \cdot ds$$

Magnetic flux and Magnetic Flux density :-

In free space $B = \mu_0 H$ B is in web/m^2 or Tesla or Gauss

$\mu_0 = 4\pi \times 10^{-7}$ H/m → permeability of free space

unit of $H = \dfrac{B}{\mu_0} = wb/m^2 \times \dfrac{m}{H} \rightarrow (A/m)$.

$$\boxed{\phi = \int_S B \cdot ds \;\; wb}$$ $\psi = \oint_S D \cdot ds = Q$

The magnetic lines are closed and do not terminate.

So Gauss law for Magnetic field is

$$\oint_S B \cdot ds = 0$$

$$\therefore \boxed{\nabla \cdot B = 0}$$

As isolated Magnetic charge does not exist. Hence

$\oint B \cdot ds = 0$ is referred as the law of conservation of Magnetic flux or Gauss law for Magnetic flux.

Hence $\boxed{\nabla \cdot B = 0}$ Maxwells fourth equation

Maxwell's equations for static fields :-

Differential (or) point form	Integral form	Remarks.
$\nabla \cdot D = \rho_v$	$\oint_S D \cdot ds = Q$	Gauss law
$\nabla \cdot B = 0$	$\oint_S B \cdot ds = 0$	Non existance of Magnetic monopole.
$\nabla \times E = \vec{0}$	$\oint_L E \cdot dL = 0$	conservation of electro static field.
$\nabla \times H = J$	$\oint_L H \cdot dL = I = \int_S J \cdot ds$	Amperi law.

Analogy between Electric and Magnetic Fields :-

	Term	Electric	Magnetic
1	Basic Laws	$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \cdot a_R$	$dB = \frac{\mu_0 I dl \times a_R}{4\pi R^2}$
		$\oint D \cdot ds = Q$	$\oint H \cdot dL = I$
2.	Force Law	$F = QE$	$F = QU \times B$
3.	Source element	dQ	$QU = Idl$
4.	Field Intensity	$E = \frac{V}{l}$ (V/m)	$H = \frac{I}{l}$ (A/m)
5.	Flux density	$D = \frac{\psi}{S}$ (C/m²)	$B = \frac{\Phi}{S}$ (wb/m²)
6.	Relationship between Fields	$D = \epsilon E$	$B = \mu H$
7.	potentials	$E = -\nabla V$	$H = -\nabla V_m \; (J=0)$
		$V = \int \frac{\rho_L dl}{4\pi\epsilon r}$	$A = \int \frac{\mu I dl}{4\pi R}$

8. Flux $\quad \psi = \oint_S D \cdot ds \qquad \phi = \oint B\, ds$

$\psi = Q = CV \qquad \phi = LI$

$I = C\frac{dv}{dt} \qquad V = L\frac{di}{dt}$

9. Energy density $\quad w_E = \frac{1}{2} D \cdot E \qquad w_m = \frac{1}{2} B \cdot H.$

10. poissons eqjoati $\quad \nabla^2 V = -\frac{\rho_v}{\epsilon} \qquad \nabla^2 A = -\mu J.$

- magnetic dipole moment $\quad m = I_b\, ds$

If there are n magnetic dipoles

$$m_{total} = \sum_{i=1}^{n\Delta v} m_i$$

magnetisation $M = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} m_i \quad A/m.$

$B = \mu_0 H$ when magnet $= 0.$

$$\boxed{B = \mu_0 H + m}$$

$M = \chi_m H$

$B = \mu_0 H + \chi_m \mu_0 H$

$= \mu_0 H (1 + \chi_m) \qquad 1 + \chi_m = \mu_R.$

$\chi_m =$ magnetic susceptibility

$\therefore \mu_R = 1 + \chi_m.$

$$\boxed{\chi_m = \mu_r - 1}$$

For ferrite Material for which $\mu_R = 50 \qquad \chi_m = 50 - 1 = 49$

and if $B = 0.05\,T$ then $B = \mu_0 \mu_R H$

$H = \frac{0.05}{\ldots} = 796\ A/m.$

Magnetic Forces :- The electric field causes a force to be exerted on a charge which may be either stationary or in motion. Steady Magnetic field is capable of exerting force only on moving charge. A magnetic field may be produced by moving charges and may exert forces on moving charges; A magnetic field can not arise from stationary charges and can not exert any force on a stationary charge.

There are 3 ways in which force due to magnetic fields can be experienced. The force can be

i) due to moving charged particle in a B field

2) on a current element in an external B field

3) between two current elements.

1. Force on A moving charge or charged particle :-

The electric force on a stationary or moving charge is given by

$$F_e = QE$$

If Q is +ve F and E have same direction.

Magnetic Force :- A charged particle in motion in a Magnetic field of flux density B is found experimentally to experience a force whose magnitude is proportional to product of magnitude of charge, its velocity v, the flux density and to the Sine of the angle between v and B. The direction of force is ⊥r to both v and B and is given by unit vector in the direction of $V \times B$

$$\boxed{F_m = Q\, V \times B}$$

The force due to two fields $F = Fe + Fm$ $F = Q(E + V \times B)$ → Lorentz force equation.

The above principle is used in Magnetron, proton paths in cyclotron Magneto hydro dynamic generator.

Problem :– A negative point charge $Q = -40\,nC$ is moving with a velocity of $6 \times 10^6\ m/s$, in a direction specified by $a_v = -0.48a_x - 0.6a_y + 0.64a_z$. Find the Magnitude of vector force exerted on moving particle by the field a) $B = 2a_x - 3a_y + 5a_z\ mT$ b) $E = 2a_x - 3a_y + 5a_z\ kV/m$

c) B and E acting together.

a) $$F = Q\,\bar{V} \times \bar{B} = -40 \times 10^{-9} \times 6 \times 10^{6} \times 10^{-3} \begin{vmatrix} a_x & a_y & a_z \\ -0.48 & -0.6 & 0.64 \\ 2 & -3 & 5 \end{vmatrix}$$

$$= 259.2\,a_x + 883.2\,a_y - 633.6\,a_z\ \mu N$$

$$|F| = 1117.442\ \mu N.$$

b) $$\bar{F} = Q\bar{E} = -40 \times 10^{-9}\left(2a_x - 3a_y + 5a_z\right)10^{3}$$

$$= -80a_x + 120a_y - 200a_z\ \mu N.$$

$$|\bar{F}| = \sqrt{(-80)^2 + (120)^2 + (-200)^2}\ a_z\ \mu N$$

$$= 246.577\ \mu N$$

c) $$\bar{F} = Q\bar{E} + Q\,\bar{V} \times \bar{B}.$$

$$= 179.2\,a_x + 1003.2\,a_y - 833.6\,a_z\ \mu N$$

$$|\bar{F}| = 1316.591\ \mu N.$$

Force on a charged particle :-

state of particle	E field	B field	combined E and B fields.
stationary	QE	—	QE
Moving	QE	$Q\,V \times B$	$Q(E + V \times B)$

Force on a differential current Element :-

The force on a charged particle moving through a steady magnetic field may be written as differential force exerted on a differential element of charge

$$dF = dQ\, V \times B$$

$$dF = \rho_v\, dv\, V \times B \qquad \boxed{J = \rho_v V}$$

$$dF = J \times B\, dv$$

$$\boxed{J\,dv = K\,ds = I\,dL}$$

$$dF = I\,dL \times B \qquad \Delta I = \frac{\Delta Q}{\Delta t}$$

$$\therefore\ F = \oint I\,dL \times B \qquad \Delta J = \rho_v \frac{\Delta s\, \Delta x}{\Delta t}$$

$$= -I \oint B \times dL \qquad \boxed{J = \rho_v \cdot V}$$

$$\boxed{F = I\, L \times B}$$

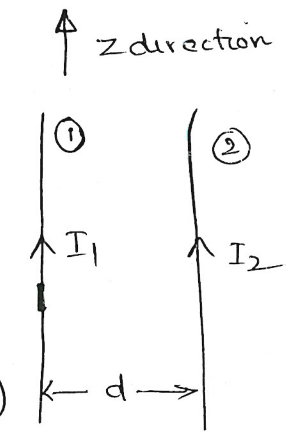
$\therefore$ The magnitude of the force is given by

$$F = BIL \sin\theta$$

where θ = angle between the direction of current flow and direction of Magnetic Flux density.

A conductor 4m long lies along y axis and with a current of 10 A in a_y direction. Find the force on conductor if $B = 0.05\, a_x$ T

$$F = IL \times B = 10.0(4a_y \times 0.05 a_x) = -2.0 a_z\ N.$$



~~Problem~~ :-

Force between Long straight conductors.

we can write

$$\bar{H}_{21} = \frac{I_1}{2\pi d}(-a_x) \quad \text{(Biot-Savart law)}$$

$$\bar{B}_{21} = \mu_0 \bar{H}_{21} = \frac{-\mu_0 I_1}{2\pi d} \bar{a}_x$$

Force experienced by unit length of conductor is

$$F_{21} = I_2 \bar{l} \times \bar{B}_{21}$$

$$= I_2 (1a_z) \times \frac{\mu_0 I_1 (-a_x)}{2\pi d}$$

$$= \frac{\mu_0 I_1 I_2}{2\pi d}(-a_y) \text{ Newtons.}$$

Problem : Determine the force per metre length b/w two long parallel wires
① A and B seperated by 5mm in Air and carryy a current of 40 Amp

a) In the same direction

b) In the opposite direction

Force experience per metre length of conducti $= \frac{\mu_0 i_1 i_2}{2\pi d}$

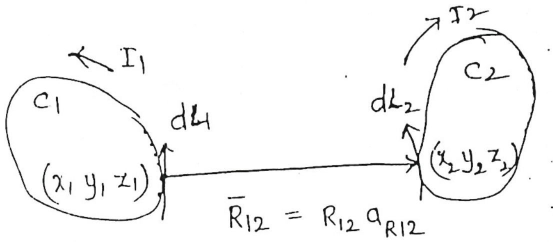
$$= \frac{4\pi \times 10^{-7} \times 40 \times 40}{2\pi \times 5 \times 10^{-3}} = 6.4 \times 10^{-3} \text{ N}.$$

The force is attractive when they carry current in same direction and repulsive when they carry current in opposite direction.

② Two wires carryig current in same directi of 5000 A and 1000 A are placed with their axis 5cm apart. calulat F.

$$F = \mu_0 i_1 i_2$$

Ampere's Force Law :- This law gives the force between a current carrying loops of thin wire. Suppose if two loops c_1 and c_2 carry currents I_1 and I_2 as shown here. It is possible to



and express the force element directly in the second element without finding Magnetic f

The Magnetic field at point 2 due to current element at point 1 is

$$dH_2 = \frac{I_1\, dL_1 \times a_{R12}}{4\pi R_{12}^2}$$

The differential force on differential current element is

$$\boxed{dF = I\, dL \times B}$$

But in this problem B is dB_2 and $I\,dL = I_2\, dL$

The differential amount of ~~of~~ differential force on element is

$$d(dF_2) = I_2\, dL_2 \times dB_2 \qquad \text{But } dB_2 = \mu_0\, dH$$

$$d(dF_2) = I_2\, dL_2 \times \mu_0 \frac{I_1\, dL_1 \times a_{R12}}{4\pi R_{12}^2}$$

$$d(dF_2) = \frac{\mu_0 I_1 I_2}{4\pi R_{12}^2}\, dL_2 \times (dL_1 \times a_{R12})$$

The total force between two filamentary circuit is ob as

$$F_2 = \frac{\mu_0 I_1 I_2}{4\pi} \oint \left[dL_2 \times \oint \frac{dL_1 \times a_{R12}}{R_{12}^2} \right.$$

$$F_2 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \left[\oint_{C_2} \frac{a_{R12} \times dL_1}{R_{12}^2} \right] \times dL_2$$

The above Expression is Magnetic field at point 2 due to the current element at point 1. From above expression

$$F_{21} = \mu_0 I_2 \oint_{C_2} dL_2 \times H_{21}$$

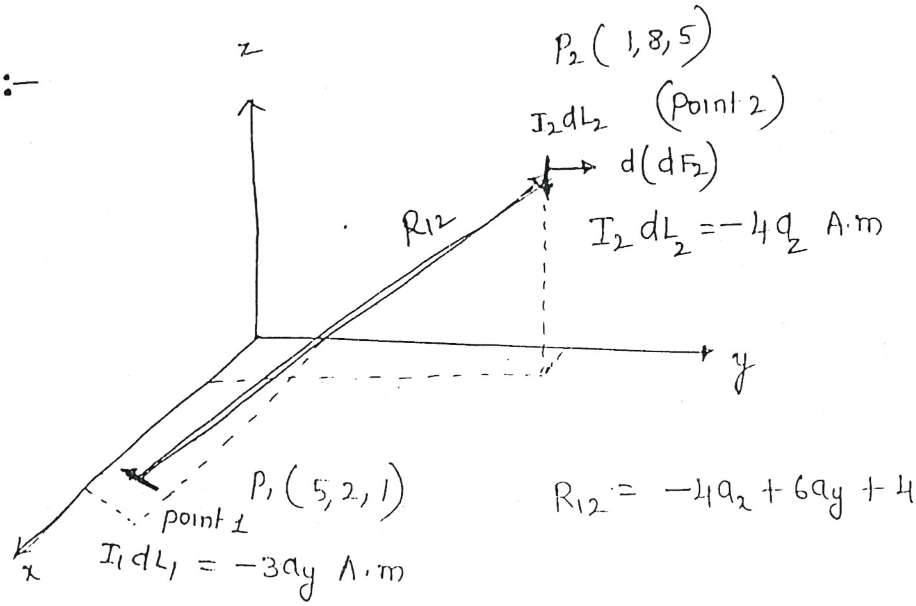
and $$H_{21} = \frac{I_1}{4\pi} \oint_{C_1} \frac{dL_1 \times a_{R12}}{R_{12}^2}$$

$$F_{21} = I_2 \oint_{C_2} dL_2 \times B_{21}$$

$$B_{21} = \mu H_{21} = \frac{\mu I_1}{4\pi} \oint_{C_1} \frac{dL_1 \times a_{R12}}{R_{12}^2}$$

In this Magnetic field case $F_{12} \neq F_{21}$

Problem :-



$R_{12} = -4a_x + 6a_y + 4a_z$

$$\therefore d(dF_2) = \frac{\mu_0 I_1 I_2}{4\pi R_{12}^2} dL_2 \times (dL_1 \times a_{R12})$$

$$= 4\pi \times 10^{-7} (-4a_z) \times \left[\frac{(-3a_y) \times (-4a_x + 6a_y + 4a_z)}{(16+36+16)^{1.5}} \right]$$

$$= 8.56\, a_y \text{ nN.}$$

Magnetization and permeability :-

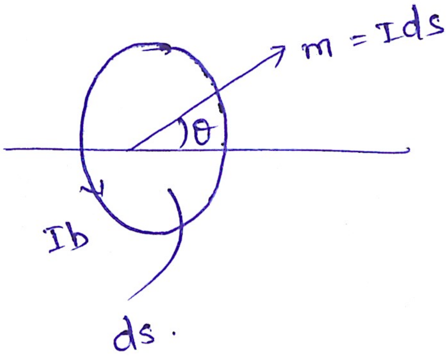
dipole moment $m = I_b\, ds$ $(\text{Amp}-m^2)$

If there are n magnetic dipoles per unit volume

$$\therefore m_{total} = \sum_{i=1}^{n\Delta v} m_i$$

Magnetisation M = magnetic dipole moment / unit volume

$$M = \lim_{\Delta v \to 0} \sum_{i=1}^{n\Delta v} m_i \qquad (A/m)$$



$$I_b = \oint M \cdot dL$$

$$I = \oint H \cdot dL$$

$$I_T = \oint \frac{B}{\mu_0} \cdot dL$$

$$I_T = I + I_b$$

$$I = I_T - I_b = \oint \left[\frac{B}{\mu_0} - M\right] \cdot dL$$

$$H = \frac{B}{\mu_0} - M$$

$$B = \mu_0 [H + M]$$

$$\boxed{M = \chi_m \bar{H}}$$

current relations :

$$I_b = \oint J_b \cdot ds$$

$$I = \oint_s J \cdot ds$$

$$I_T = \oint_s J_T \cdot ds$$

$$\boxed{\begin{aligned} \nabla \times M &= J_b \\ \nabla \times H &= J \\ \nabla \times \frac{B}{\mu_0} &= J_T \end{aligned}}$$

Electromagnetic Induction

Faraday's Law : when ever there is a rate of change of flux e.m.f will be produced in a conductor. $e = -N \frac{d\phi}{dt}$ Volts.

Problem : A coil of 100 turns is linked by a flux of 20 mwb. If the flux is reversed in a time of 2 ms. calculate e.m.f

$$e.m.f = N \frac{d\phi}{dt} = 100 \; \frac{20-(-20)}{2 \times 10^{-3}} = \boxed{2000\,V}$$

Direction : Direction of Induced EMF will be given by two Laws

i) Lenz's Law : An induced current will flow in such a direction so as to oppose the cause that produces it.

ii) Flemings Right hand rule :

Fore finger → Direction of Magnetic field

Thumb → Motion of conductor

Middle finger → Direction of Induced current.

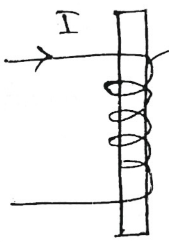
Induced EMF.

- Statically Induced EMF
  (conductor is stationary
  Magnetic field is moving)
  EX : Transformer
  - Self Induced EMF
  - Mutually Induced EMF
- Dynamically Induced EMF
  (stationary magnetic field
  conductor is moving)
  EX : DC generators
  DC Motors.

Self Inductance (L) :- The property of a coil that opposes any chang in the amount of current flowing through it is called Self Inductan or Inductance :: The Inductance of the coil depends on

1) shape and No of turns
2) μ_r of the material surrounding the coil
3) The speed with which the magnetic field changes.

Expressions :-



N turns

$$e = \frac{N\,d\phi}{dt}$$

$$e \propto \frac{dI}{dt}$$

$$e = \frac{d}{dt}(N\phi) \quad -(1)$$

$$e = L\frac{di}{dt} \quad -(2)$$

Equating 1 and 2 $\quad LI = N\phi$

$$\therefore \boxed{L = \frac{N\phi}{I}}$$

Magnetomotive force (m.m.f) :- It is equal to the product of current and no of turns of the coil.

$$m.m.f = NI \ (AT)$$

Reluctance (S) : It is the property to oppose the creation of magnetic flux.

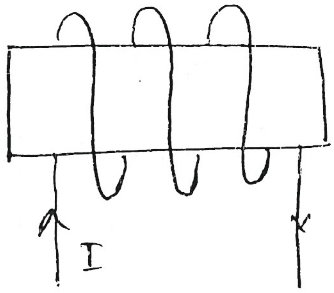
$$\boxed{S = \frac{l}{\mu_0\mu_r A}} \ wb/AT.$$

$$mmf = \frac{flux}{Reluctance}$$

permeance : It is reciprocal of reluctance.

$$permeance = \frac{1}{reluctance} = \frac{a\,\mu_0\mu_r}{l}.$$

Inductance of a Solenoid :-



$$L = N\frac{d\phi}{dI}$$

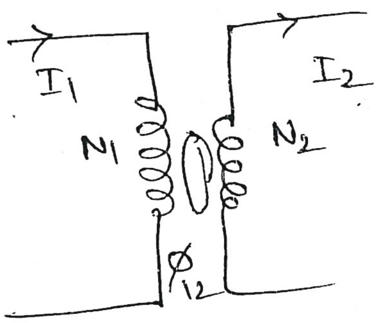
$$\phi = \frac{m.m.f}{reluctance} = \frac{NI}{l/\mu_0\mu_r A}$$

$$\frac{d\phi}{dI} = \frac{N\mu_0\mu_r A}{l}$$

$$L = \frac{N.N\mu_0\mu_r A}{l} = \frac{N^2}{S}$$

$$\boxed{L = \frac{N^2}{S}}$$

Mutual Inductance :- The two coils so arranged that a change of current in one coil causes an e.m.f to be induced in other are said to have Mutual Inductance.



$$\phi_{12} = \frac{mmf}{S} = \frac{N_1 I_1}{l/\mu_0\mu_r A}$$

$$\frac{\phi_{12}}{I_1} = \frac{N_1\mu_0\mu_r A}{l}$$

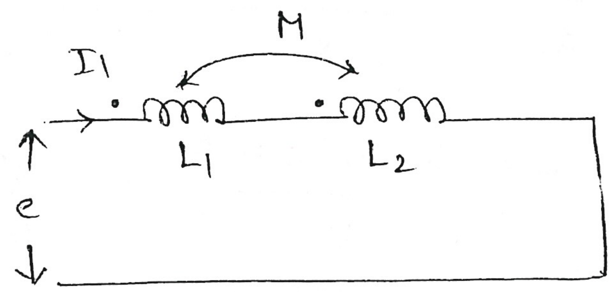
$$M = \frac{N_2\phi_{12}}{I_1} = \frac{N_1 N_2\mu_0\mu_r A}{l}$$

$$\boxed{M = \frac{N_1 N_2}{S}}$$

Coefficient of coupling (K) :- The coefficient of coupling between two coils is defined as the fraction of magnetic flux produced by the current in one coil that links the other coil.

$$\boxed{M = K\sqrt{L_1 L_2}}$$

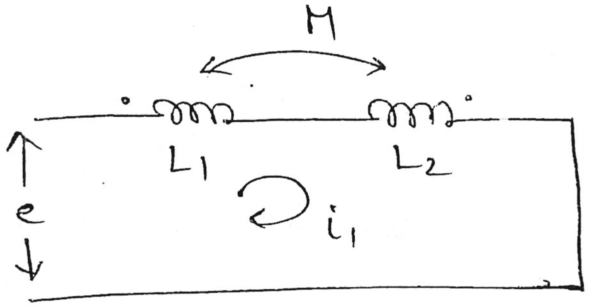
Inductances in Series Aiding :



$$e = L_1\frac{di_1}{dt} + L_2\frac{di_1}{dt} + M\frac{di_1}{dt} + M\frac{di_1}{dt}$$

$$e = (L_1 + L_2 + 2M)\frac{di_1}{dt}.$$

Inductances in Series opposing :



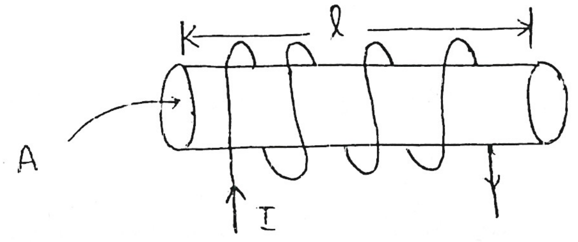
$$e = L_1\frac{di_1}{dt} + L_2\frac{di_1}{dt} - M\frac{di_1}{dt} - M\frac{di_1}{dt}$$

$$e = (L_1 + L_2 - 2M)\frac{di_1}{dt}.$$

Self Inductance (L) : Inductance or Self Inductance is the ratio of total flux linkages to the current which they link.

$$\therefore \quad L = \frac{N\phi}{I} \quad \text{henry (H) or web-turn/Ampere.}$$

Self Inductance of a Long Solenoid :-



Air core Solenoid

consider a long air core solenoid of length 'l' metre and uniform cross sectional area 'A' m^2 and let 'n' be the no of turns per metre. If I is the current flowing through solenoid we have

$$B = \mu_0 H \quad \text{and} \quad H = \frac{nI}{l}$$

$$\therefore \quad B = \frac{\mu_0 n I}{l} \; wb/m^2 \quad \text{per one metre}$$

$\therefore$ Magnetic flux through each turn $\phi = BA = \mu_0 n I A$ weber

If N is the total No of turns then the total Magnetic flux of all turns of solenoid is

$$\phi_T = \mu_0 n I A \times N \quad \text{weber turns}$$

$$\phi_T = \mu_0 n I A \times nl \qquad (\because N = nl)$$

$$\phi_T = \mu_0 n^2 I A l$$

By Difinition $L = \frac{N\phi}{I} = \frac{N \cdot \mu_0 n^2 \cancel{I} A l}{\cancel{I}}$

$$= \mu_0 n^2 A l$$

$$L = \mu_0 \left(\frac{N}{l}\right)^2 A l = \boxed{\frac{\mu_0 N^2 A}{l}} \text{ henrys.}$$

Problem : Find the Inductance of a solenoid of 2800 turns wound uniformly over a length 0.6 m on a cylindrical paper tube 4 cm in diameter. The medium is air.

Solution : $N = 2800$ $\quad l = 0.6\,m$ $\quad d = 4\,cm = 4\times10^{-2}\,m$

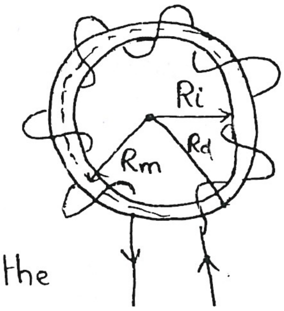
$$A = \frac{\pi d^2}{4} = 1.256\times10^{-3}\ sq.m$$

$$\therefore \quad L = \frac{\mu_0 N^2 A}{l} = \frac{4\pi\times10^{-7}(2800)^2\times1.256\times10^{-3}}{0.6}$$

$$\boxed{L = 20.62\,mH}$$

Self Inductance of a Toroid :-

If a long solenoid is bent in the form of a ring it becomes toroidal solenoid or toroid.



when a toroid is provided with uniformly spaced turns of 'N'

The average flux density inside the toroid is given by

$$B = \mu_0 \frac{NI}{l_m}$$

Here l_m = mean length of path of flux.

$$l_m = 2\pi R_m$$

$$B = \mu_0 \frac{NI}{2\pi R_m}$$

For N turns the total flux $\phi_T = N\phi = NBA$

$$\phi_T = N \cdot \left(\mu_0 \frac{NI}{2\pi R_m} \right) \pi r^2$$

$$= \frac{\mu_0 N^2 I \cancel{\pi} r^2}{2 \cancel{\pi} R_m}$$

$$\phi_T = \mu_0 \frac{N^2 r^2}{2 R_m} I$$

$\therefore$ The Inductance of the toroid is

$$L = \frac{\phi_T}{I} = \frac{\mu_0 N^2 r^2}{2 R_m}$$

(r = mean radius of coil)

Problem : A toroidal coil of 500 turns is wound on a steel ring of 0.5m mean dia and 2×10^{-2} m^2 cross sectional area. An excitation of $4000 A m^{-1}$ produces a flux density of 1 Tesla. Find the Inductance of the coil.

solution :

$$L = \frac{N\phi}{I} = \frac{NBA}{I} = \frac{N}{I}\mu H A$$

$$= \frac{N}{\cancel{I}}\mu \frac{N\cancel{I}}{l} A$$

$$\mu = \frac{B}{H} = \frac{1}{4000} = 0.25 \times 10^{-3} \ H/m$$

$$= \frac{\mu N^2 A}{l}$$

$$\therefore \ L = \frac{(500)^2 \times 0.25 \times 10^{-3} \times 2 \times 10^{-3}}{2\pi \times 0.25} = \boxed{80mH.}$$

If 10mm long gap is cut in the ring find the current required to maintain the flux density at 1Tesla. Also find the inductance under these new conditions. Neglect all the Leakage and fringing.

$$NI = H_i l_i + H_a l_a$$

$$NI = \frac{B_i}{\mu_i} l_i + \frac{B_a}{\mu_a} l_a$$

$$l_i = 0.5\pi m$$

$$l_a = 10 \times 10^{-3} m$$

$$NI = \frac{1 \times 0.5\pi}{0.25 \times 10^{-3}} + \frac{1 \times 10 \times 10^{-3}}{4\pi \times 10^{-7}}$$

Solving $I = 28.4$ Amp.

$$L = \frac{N\phi}{I} = \frac{NBA}{I}$$

$$= \frac{500 \times 1 \times 2 \times 10^{-3}}{28.4}$$

$$= 35mH.$$

Energy stored in a Magnetic Field :- In order to establish magnetic field around a coil, energy is required but no energy is needed to maintain it. This energy is stored in the magnetic field, if the current is increased from 0 to I with the potential difference across inductor equal to V then the source supplying power is VI

Energy supplied by the source is $VI\,dt$

Let dw be the workdone to increase the current by dI. By the law of conservation of Energy work done is equal to Energy st

$$dw = VI\,dt$$

$$= I\,L\frac{di}{dt}\,dt.$$

$$dw = LI\,di$$

$$\therefore \quad w = L\int_0^I I\,di$$

$$w = \frac{1}{2}LI^2$$

$$\boxed{\begin{aligned} \therefore \quad w &= \tfrac{1}{2}LI^2 \text{ Joules} \\ \text{and} \quad L &= \frac{N\phi}{I}. \end{aligned}}$$

Energy density in a Magnetic Field :- Energy density per unit volume is called Energy density.

we know that Energy stored $= \frac{1}{2}LI^2 = \frac{1}{2}\left(\frac{N^2\mu_0\mu_r A}{l}\right)I^2$

$$= \frac{1}{2}\mu_0\mu_r\left(\frac{NI}{l}\right)^2 la$$

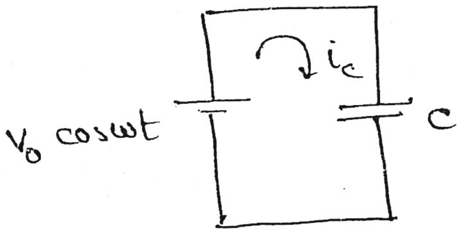
$$\therefore \frac{\text{Energy stored}}{\text{Volume}} = \frac{1}{2}\mu H^2 \text{ Joules}/m^3.$$

$$w_d = \frac{1}{2}\mu H^2 = \frac{1}{2} BH = \frac{B^2}{2\mu} \ J/m^3.$$

$$w_d = \frac{1}{2}\ \bar{B}\cdot\bar{H}\ \ J/m^3$$

$$\therefore W = \frac{1}{2}\int_v B\cdot H\, dv = \frac{1}{2}\int \mu H^2\, dv$$

Show that In a capacitor conduction current is equal to Displacement current.



The conduction current

$$i_c = C\frac{dv}{dt}$$

$$= C\cdot\frac{d}{dt}\left(V_0 \cos\omega t\right) = \omega V_0 C \sin\omega t \quad \text{———} \ (1)$$

The displacement current density

$$i_d = \epsilon \frac{\partial E}{\partial t} = \epsilon\frac{\partial}{\partial t}\left(\frac{V}{d}\right)$$

$$= \frac{\epsilon}{d}\frac{\partial}{\partial t}(V)$$

$$\frac{\partial D}{\partial t} = \frac{\epsilon}{d}\omega V_0 \sin\omega t = J_d$$

$\therefore$ The displacement current is

$$I_d = J_d\cdot A = \frac{\epsilon}{d}\omega V_0 \sin\omega t\cdot A \quad \text{———} \ (2)$$

Maxwell's equations.

we Know that Maxwell's equations for steady electric and Magnetic field. steady electric field is due to charge at rest and steady magnetic field is due to steady current. The equations are summarised as

$$\nabla \times E = 0 \qquad \oint E \cdot dL = 0 \qquad \text{— (1)}$$

$$\nabla \cdot D = \rho \qquad \oint D \cdot ds = \int \rho_v \, dv = Q \qquad \text{— (2)}$$

$$\nabla \times H = J \qquad \oint H \cdot dL = \int J \cdot ds \qquad \text{— (3)}$$

$$\nabla \cdot B = 0 \qquad \oint B \cdot ds = 0 \qquad \text{— (4)}$$

Equation of continuity for steady currents is $\nabla \cdot J = 0$ $\oint J \cdot ds = 0$

Development of above equations for Time varying fields :-

Faraday's law : It states that electromotive force around a closed path is equal to the negative of time rate of change of Magnetic flux enclosed by the path.

$$\therefore \; e.m.f = -\frac{d\phi}{dt} \; V$$

For n turns
$$e.m.f = -\frac{N d\phi}{dt}$$

$$\therefore \; e.m.f = \oint E \cdot dL = -\frac{d}{dt}\int_S B \cdot ds$$

$$\int_S (\nabla \times E) \cdot ds = -\int_S \frac{\partial B}{\partial t} \cdot ds$$

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t}}$$

If B is not a function of time then $\nabla \times E = 0$.

Equation of continuity for Time Varying Fields :-

According to the conservation of charge concept we have

$$\oint J \cdot ds = -\frac{d}{dt}\int_v \rho \, dv$$

$$\oint J \cdot ds = -\int \frac{\partial \rho}{\partial t} dv$$

By divergence theorem we have

$$\int \nabla \cdot J \, dv = -\int \frac{\partial \rho}{\partial t} dv$$

$$\therefore \boxed{\nabla \cdot J = -\frac{\partial \rho}{\partial t}}$$

Inconsistency of Amper's Law :-

We know that $\nabla \times H = J$ Taking divergence on both sides

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J$$

But according to vector calculus LHS = 0

$$\therefore \nabla \cdot J = 0 \quad \text{------} (1)$$

But equation of continuity is $\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}$ — (2)

So 1 and 2 gives different answers which are not correct. Hence we take

$$\nabla \times H = J + G$$

$$\nabla \cdot (\nabla \times H) = \nabla \cdot (J + G)$$

$$\nabla \cdot J + \nabla \cdot G = 0$$

$$\therefore \nabla \cdot G = \frac{\partial \rho_v}{\partial t}$$

Replacing ρ_v by $\nabla \cdot D$ we have

$$\nabla \cdot G = \frac{\partial}{\partial t}(\nabla \cdot D) = \nabla \cdot \frac{\partial D}{\partial t} \quad \therefore \quad G = \frac{\partial D}{\partial t}$$

Hence the equation after modification is

$$\boxed{\nabla \times H = J + \frac{\partial D}{\partial t}}$$

$$\nabla \times H = J + J_d \quad \because \boxed{J_d = \frac{\partial D}{\partial t}}$$

If the volume charge density $\rho_v = 0$ then

$$\boxed{\nabla \times H = \frac{\partial D}{\partial t} \qquad \nabla \times E = -\frac{\partial B}{\partial t}}$$

Maxwell's equations for Time varying fields :-

point form

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\dot{B}$$

$$\nabla \times H = \dot{D} + J$$

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

equation of continuity $\nabla \cdot J = -\dot{\rho}$

integral form.

$$\oint E \cdot dL = -\int \dot{B} \cdot ds$$

$$\oint H \cdot dl = \int (\dot{D} + J) \cdot ds$$

$$\oint D \cdot ds = \int_v \rho_v \, dv$$

$$\oint B \cdot ds = 0$$

$$\oint J \cdot ds = -\int \dot{\rho} \, dv$$

Word statement of Field equations :-

$\nabla \times E = -\dot{B}$

1. The electromotive force around a closed path is equal to the time derivative of Magnetic displacement through any surface bounded by the path

(or)

2. The electric voltage around a closed path is equal to the Magnetic current through the path.

$$\nabla \times H = J + \dot{D}$$

2. The magnetomotive force around a closed path is equal to the conduction current plus the time derivative of electric displacement through any surface bounded by the path.

$$\nabla \cdot D = \rho_v$$

3. The total electric displacement through the surface enclosing a volume is equal to the charge within the volume.

4. The net Magnetic flux emerging through any closed surface is zero.

$$\nabla \cdot B = 0$$

Boundary conditions :-

$$E_{tan1} = E_{tan2}$$

$$H_{tan1} = H_{tan2}$$

$$B_{N1} = B_{N2}$$

$$D_{N1} - D_{N_2} = \rho_s$$

If we do not have nice materials to work then

$$D = \epsilon_0 E + P$$

$$B = \mu_0 H + \mu_0 M$$

For linear Materials

$$P = \chi_e \epsilon_0 E$$

$$M = \chi_m H$$

The Lorentz force equation per unit volume is

$$F = \rho_v (E + V \times B)$$

For perfect conductor : $\sigma = \infty$ $J = \text{finite}$

Hence $\boxed{E = 0 \quad H = 0 \quad J = 0}$

Conditions at a Boundary Surface :-

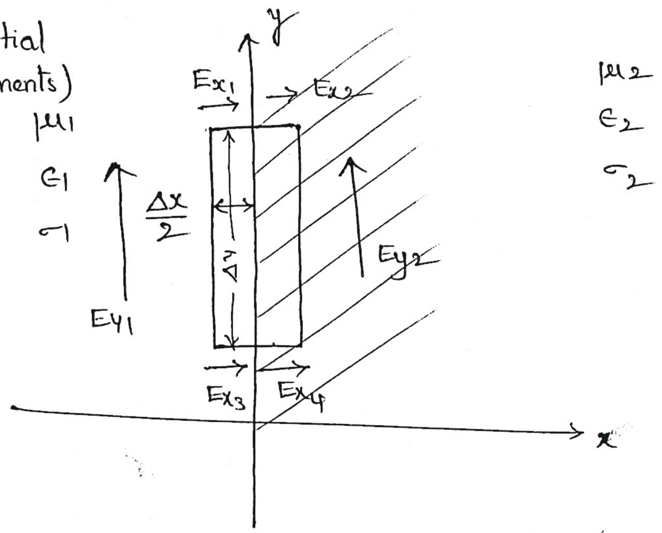
a) The tangential component of E is continuous at the surface. $E_{tan1} = E_{tan2}$

b) The tangential component of H is continuous across the surface except at the surface of a perfect conductor. At the surface of a perfect conductor H_{tan} is discontinous by an amount equal to the surface current per unit width.

c) B_{normal} is continous at the surface of discontinuity.

d) The Normal component of D is continous if there is no surface charge density. otherwise D is discontinous by an amount equal to surface charge density.

Proof :- (Tangential components)



If the surface of discontinuity is at $x = 0$ (plane)

By applying $\oint E \cdot dL = -\int_S \dot{B} \cdot ds$

For the above rectangle

$$E_{y_2} \Delta y - E_{x_2} \cdot \frac{\Delta x}{2} - E_{x_1} \frac{\Delta x}{2} - E_{y_1} \Delta y + E_{x_3} \frac{\Delta x}{2} + E_{x_4} \frac{\Delta x}{2} = -\dot{B}_z \Delta x \Delta y$$

where B_z = average flux density through $\Delta x \Delta y$

For the sharp discontinuity $\Delta x \Rightarrow 0$ and Assuming B is always finite we have

$$E_{y_2} \Delta y - E_{y_1} \Delta y = 0$$

$$\boxed{E_{y_2} = E_{y_1} \quad \text{i.e} \quad E_{tan2} = E_{tan1}}$$

||ly for H we have

$$H_{y_2} \Delta y - H_{x_2} \frac{\Delta x}{2} - H_{x_1} \frac{\Delta x}{2} - H_{y_1} \Delta y + H_{x_3} \frac{\Delta x}{2} + H_{x_4} \Delta x/2$$
$$= (\dot{D}_z + J_z) \Delta x \Delta y$$

considering both $\dot{D}_z$ and J_z finite

we have

$$H_{y_2} \Delta y - H_{y_1} \Delta y = 0 \qquad \boxed{\begin{array}{l} H_{y_2} = H_{y_1} \\ H_{tan2} = H_{tan1} \end{array}}$$

when current sheet is placed at the Boundary :-

$$\lim_{\Delta x \to 0} J \Delta x = J_s \quad A/m$$

$$H_{y_2} \Delta y - H_{y_1} \Delta y = J_{s_z} \Delta y$$

$$H_{y_1} = H_{y_2} - J_{s_z}$$

If Electric field is zero within a conductor then Magnetic field is also zero.

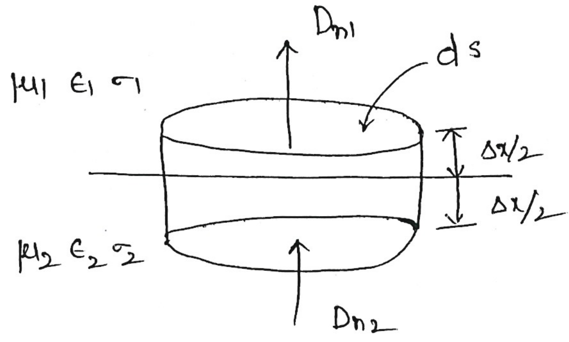
$$\therefore H_{y_1} = -J_{s_z}$$

This equation states that current per unit width along the surface of a conductor is equal to H just outside the surface.

$$\therefore J_s = \hat{n} \times H$$

where $\hat{n}$ is unit vector along outward Normal.

Normal components :-



By applying Gauss law to the pill box we have

$$\oint_S D \cdot ds = \int_v \rho \, dv$$

$$D_{n1} \, ds - D_{n2} \, ds + \psi_{edge} = \rho \, \Delta x \, ds$$

For finite values of displacement

$$D_{n1} ds - D_{n2} ds = 0 \quad \text{as} \quad \Delta x \to 0$$

$$\therefore D_{n1} = D_{n2}$$

If the layer of surface charge has a surface charge density ρ_s

$$D_{n1} - D_{n2} = \rho_s$$

If the Medium 2 is metallic conductor then $D_{n2} = 0$

$$\boxed{D_{n1} = \rho_s}$$

The Normal component of displacement density in the dielectric is equal to the surface charge density on the conductor.

In the case of B we have

$$\boxed{B_{n1} = B_{n2}}$$

The normal component of Magnetic flux density is always continuous across a boundary surface.

Prove that the displacement current is equal to conduction current in capacitor :-

Point Electrostatic	Integral form	Time Varying Point	Integral form	Free space electrostatic Point	Integral form
$\nabla \cdot D = \rho_v$	$\psi = \int D ds$	$\nabla \cdot D = \rho_v$	$\psi = \int D ds$	$\nabla \cdot D = 0$	$\psi = \int D ds$
$\nabla \times E = 0$	$\oint E dl = 0$	$\nabla \times E = \frac{-\partial B}{\partial t}$	$\int E dl = \frac{-\partial}{\partial t} \int B ds$	$\nabla \times H = 0$	$\oint H dl = 0$
$\nabla \times H = J$	$\oint H dl = I_{enc}$	$\nabla \times H = J + \frac{\partial D}{\partial t}$	$\oint H dl = \int J dl + \int \frac{\partial D}{\partial t}$	$\nabla \cdot B = 0$	$\int B ds = 0$
$\nabla \cdot B = 0$	$\oint B ds = 0$	$\nabla \cdot B = 0$	$\oint B ds = 0$	$\nabla \times E = 0$	$\oint E dl = 0$

Time Varying Free Space	Integral form	Good conductor electrostatic	Integral form	Good conductor Time Varying	Integral form
$\nabla \cdot D = 0$	$\int D ds = 0$	$\nabla \cdot D = 0$	$\psi = \int D ds$	$\nabla \times E = \frac{-\partial B}{\partial t}$	$\int E dl = -\frac{\partial}{\partial t} \int B ds$
$\nabla \times E = \frac{-\partial B}{\partial t}$	$\int E dl = \frac{-\partial}{\partial t} \int B ds$	$\nabla \times E = 0$	$\int E dl = 0$	$\nabla \times H = J$	$\oint H dl = I_{enc}$
$\nabla \times H = \frac{\partial D}{\partial t}$	$\oint H dl = D$	$\nabla \times H = J$	$\oint H dl = I_{enc}$	$\nabla \cdot D = 0$	$\psi = \int D ds$
$\nabla \cdot B = 0$	$\oint B ds = 0$	$\nabla \cdot B = 0$	$\oint B ds = 0$	$\nabla \cdot B = 0$	$\oint B ds = 0$

Phasor form:

$$\nabla \cdot D = \rho_v$$

$$\nabla \times H = E(\sigma + j\omega\varepsilon)$$

$$\nabla \times E = -j\omega\mu H$$

$$\nabla \cdot B = 0$$

Electromagnetic waves

The Maxwell's equations for time varying fields are written in point form as

$$\nabla \cdot \bar{D} = \rho_v \quad —(1)$$

$$\nabla \times \bar{E} = -\dot{\bar{B}} = -\mu\dot{\bar{H}} \quad —(2)$$

$$\nabla \times \bar{H} = J + \dot{\bar{D}} \quad —(3)$$

$$\nabla \cdot B = 0 \quad —(4)$$

The relations of Medium characteristics are

$$\boxed{\begin{aligned}\bar{D} &= \epsilon\bar{E}\\ \bar{B} &= \mu\bar{H}\\ J &= \sigma\bar{E}\end{aligned}}$$

Homogenious Medium : It is the medium for which σ, ϵ, μ are const. It is isotropic if ϵ is a scalar quantity.

Wave Equations in Free space :- Free space is a medium in which charge densities and conduction currents are zero. ($\rho_v = 0, J = 0$) It is also called as Lossless Medium as ($\sigma = 0$) conductivity is zero.

The Maxwell's equations are

$$\nabla \cdot \bar{D} = 0 \quad —(1)$$

$$\nabla \times E = -\dot{B} = -\mu\dot{H} \quad —(2)$$

$$\nabla \times H = \dot{D} = \epsilon\dot{E} \quad —(3)$$

$$\nabla \cdot B = 0 \quad —(4)$$

Taking the 2nd equation and applying curling on both sides

$$\nabla \times \nabla \times E = -\mu\, \nabla \times \dot{H}$$

From vector calculus $\nabla \times \nabla \times E \equiv \nabla(\nabla \cdot E) - \nabla^2 E$

$$\therefore \quad \nabla \cdot (\nabla \cdot E) - \nabla^2 E = -\mu\, \nabla \times \dot{H}$$

From (4) $\nabla \cdot E = 0$ and From (3) $\nabla \times \dot{H} = \epsilon\ddot{E}$

$$\cancel{0} - \nabla^2 E = \cancel{-}\mu\epsilon\ddot{E}$$

$$\boxed{\nabla^2 E = \mu\epsilon \frac{\partial^2 E}{\partial t^2}} \quad \text{— E wave equation}$$

Illy

$$\boxed{\nabla^2 H = \mu\epsilon \frac{\partial^2 H}{\partial t^2}} \quad \text{— H wave equation}$$

The wave equation is $\nabla^2 E = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$ Mathematically

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

The above equation represents waves in three directions x, y, z whic is not easy to handle. Hence the concept of uniform plane wave comes into picture.

Uniform plane wave (upw) :- If E and H are independent of y, z other than direction of propagation it is called as upw. If the wave is propagating in x direction then $\frac{\partial^2 E}{\partial y^2} = \frac{\partial^2 E}{\partial z^2} = 0$

Hence $$\frac{\partial^2 E}{\partial x^2} = \mu\epsilon \frac{\partial^2 E}{\partial y^2}$$ — upw in 'x' direction.

Property of upw : upw should not have the components of E and in the direction of propagation. If upw is in x direction $E_x = H_x = 0$

The equations are
$$\frac{\partial^2 E_y}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_y}{\partial t^2}$$
$$\frac{\partial^2 E_z}{\partial x^2} = \mu\epsilon \frac{\partial^2 E_z}{\partial t^2}$$

Proof : In free i.e in which charge density $\rho_v = 0$

$$\nabla \cdot \bar{D} = 0 \quad \text{i.e} \quad \nabla \cdot \bar{E} = 0$$

i.e
$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

For upw in 'x' direction $\frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z} = 0$

$$\therefore \Rightarrow \frac{\partial E_x}{\partial x} = 0$$

There is no varition of E_x in x direction

So $\frac{\partial E_x}{\partial x} = 0$ when $E_x = 0$ (or)

E_x = const or increasing uniformly with time

Hence upw propagating in 'x' direction has no components of E_x and H_x. Hence upw is transverse and have components of E and H only in $\perp$r direction of propagation. Hence upw is TEM wave.

Relation between E and H in upw :-

Prove that intrinsic Impedance $(\eta_0 = 377\Omega)$:-

Derive the expression $\eta_0 = \frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi = 377\Omega$:-

For a upw propagating in 'x' direction the Maxwell equation can be written as (From 2nd equation)

$$\nabla \times E = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & E_y & E_z \end{vmatrix} = -\mu \dot{H} = -\mu\left(\frac{\partial H_y}{\partial t} a_y + \frac{\partial H_z}{\partial t} a_z\right)$$

Expanding the above determinant

$$-\frac{\partial E_z}{\partial x} a_y + \frac{\partial E_y}{\partial x} a_z = -\mu \frac{\partial H_y}{\partial t} a_y - \mu \frac{\partial H_z}{\partial t} a_z$$

Equating the similar terms from both sides we have

$$\frac{\partial E_z}{\partial x} = \mu \frac{\partial H_y}{\partial t} \quad -(1)$$

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t} \quad -(2)$$

lly From (3rd equation of Maxwell)

$$-\frac{\partial H_z}{\partial x} = \epsilon \frac{\partial E_y}{\partial t} \quad -(3)$$

$$\frac{\partial H_y}{\partial x} = \epsilon \frac{\partial E_z}{\partial t} \quad -(4)$$

From any above equations we can develop the relation b/w E and H.

Let us take 2nd equation and Assume the solution

$$E_y = f(x - V_0 t) \quad \text{here } V_0 = \frac{1}{\sqrt{\mu \epsilon}} \text{ velocity of wave}$$

$$\frac{\partial E_y}{\partial t} = \frac{\partial f}{\partial (x - V_0 t)} (-V_0) = -V_0 f'$$

Substituting in 2nd equation

$$\cancel{-} V_0 \epsilon f' = \cancel{-} \frac{\partial H_z}{\partial x}$$

integrating on both sides w.r. to 'x'

$$\int \frac{\partial H_z}{\partial x} dx = \int V_0 \epsilon f'$$

$$H_z = V_0 \epsilon f \quad \text{and putting } f = E_y$$

$$H_z = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \epsilon_0 \epsilon_r \cdot E_y \quad (\because \epsilon_r = 1)$$

$$\eta_0 = \frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}} = 377\,\Omega.$$

Hence the Intrinsic impedance of freespace is resistive.

As the wave is propagating in 'x' direction $E_x = 0 \; H_x = 0$

$$\therefore (\bar{E} \cdot \bar{H}) = (E_x a_x + E_y a_y + E_z a_z) \cdot (H_x a_x + H_y a_y + H_z a_z)$$

$$= E_y H_y + E_z H_z$$

$$= \eta_0 H_y H_z - \eta_0 H_y H_z = 0$$

$\therefore \bar{E} \cdot \bar{H} = 0$ Hence upw is TEM wave.

Wave equations for a conducting Medium :-

For the regions where the conductivity is not zero and conduction currents exits then the equations will have different solutions.

we have
$$\nabla \times H = \epsilon \dot{E} + J \quad -(1)$$
$$\nabla \times E = -\mu \dot{H} \quad -(2)$$

The conduction current density is given by $J = \sigma E$ where (σ = mhos/m)

Then the equation (1) becomes
$$\nabla \times H = \epsilon \dot{E} + \sigma E$$

Taking curl on both sides of (2)
$$\nabla \times \nabla \times E = -\mu \, \nabla \times \dot{H}$$
$$\nabla \nabla \cdot E - \nabla^2 E = -\mu\epsilon \ddot{E} - \mu\sigma \dot{E} \quad -(3)$$

Since there is no netcharge in a conductor $\nabla \cdot E = 0$

The equation (3) becomes
$$\boxed{\nabla^2 E - \mu\epsilon \ddot{E} - \mu\sigma \dot{E} = 0}$$

Equation for H : Taking the Eq (1) and apply curling on both sides
$$\nabla \times \nabla \times H = \epsilon \, \nabla \times \dot{E} + \sigma \, \nabla \times E$$
$$\nabla \nabla \cdot H - \nabla^2 H = -\mu\epsilon \ddot{H} - \mu\sigma \dot{H}$$

Since $\nabla \cdot H = 0$

the above equation becomes
$$\boxed{\nabla^2 H - \mu\epsilon \ddot{H} - \mu\sigma \dot{H} = 0}$$

These wave equations are called as Helmholtz or dissipative wave equations.

Sinusoidal Time Variation :- As the most generators produce voltage and currents and hence Electric and Magnetic fields which vary sinusoidally with time, it is customary in most problems to assume sinusoidal time variations.

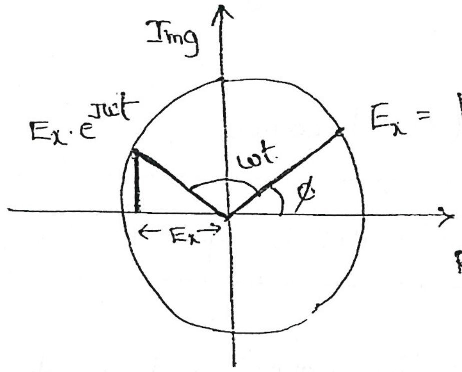
phasor Notation :- Consider Electric field strength vector as an example. The time varying field $\tilde{E}(r,t)$ may be expressed in terms of correspon phasor quantity $E(r)$ as

$$\tilde{E}(r,t) = Re\left\{ E(r)\, e^{J\omega t} \right\}$$

~ has been placed over time varying quantity to distinguish it from phasor quantity.

Then the phasor E_x is defined by

$$\tilde{E}_x(r,t) = Re\left\{ E_x(r)\, e^{J\omega t} \right\}$$



In the diagram multiplication by $e^{J\omega t}$ results in a rotation through the angle ωt measured from the angle ϕ

$$\tilde{E}_x = Re\left\{ |E_x|\, e^{J\phi} \cdot e^{J\omega t} \right\}$$

$$= |E_x| \cos(\omega t + \phi)$$

Taking the real part is same as taking projection on real axis and this projection varies sinusoidally with time.

Maxwell's equations using phasor notation :-

For sinusoidal time variations the first equation is

$$\nabla \times \tilde{H} = \frac{\partial \tilde{D}}{\partial t} + \tilde{J}$$

$$\nabla \times \text{Re}\left\{ H e^{j\omega t} \right\} = \frac{\partial}{\partial t}\left(\text{Re}\left(D \cdot e^{j\omega t}\right)\right) + \text{Re}\left\{ J \cdot e^{j\omega t} \right\}$$

By taking real part and changing order of differentiation

$$\text{Re}\left\{ (\nabla \times H - j\omega D - J)\, e^{j\omega t} \right\} = 0$$

If the above relation is valid for all t

$$\nabla \times H = j\omega D + J$$

* Note : (The phasor equation may be derived from time varying equation by replacing each time varying quantity with a phasor quantity and each time derivative with a $j\omega$ factor)

Maxwells equations in phasor form ;

$$\nabla \times H = j\omega D + J \qquad \oint H \cdot dL = \int (j\omega D + J) \cdot ds$$

$$\nabla \times E = -j\omega B \qquad \oint E \cdot dL = -\int j\omega B \cdot ds$$

$$\nabla \cdot D = \rho \qquad \oint D \cdot ds = \int \rho \, dv$$

$$\nabla \cdot B = 0 \qquad \oint B \cdot ds = 0$$

The equation of continuity is given by

$$\nabla \cdot J = -j\omega \rho \qquad \oint J \cdot ds = -\int j\omega \rho \, dv$$

The constructive relations are

$$D = \epsilon E, \qquad B = \mu H, \qquad J = \sigma E$$

The wave equation in a lossless medium

$$\nabla^2 E = -\omega^2 \mu\epsilon E \rightarrow \text{Vector Helmholtz equation}$$

The wave equation in a conducting medium is

$$\nabla^2 E + (\omega^2 \mu\epsilon - j\omega\mu\sigma) E = 0$$

<u>wave propagation in a Lossless medium</u> (Freespace) $(\sigma = 0)$

For a upw case in which there is no variation in y or z the wave equation in phasor form is

$$\frac{\partial^2 E}{\partial x^2} = -\omega^2 \mu\epsilon E \quad \text{or} \quad \frac{\partial^2 E}{\partial x^2} = -\beta^2 E$$

where $\beta = \omega\sqrt{\mu\epsilon}$

considering E_y component

$$E_y = C_1 \cdot e^{-j\beta x} + C_2 e^{+j\beta x}$$ (where C_1 and C_2 are cc

For time varying field

$$\tilde{E}_y(x,t) = Re\left\{ E_y(x) \cdot e^{j\omega t} \right\}$$

$$= Re\left\{ C_1 \cdot e^{-j\beta x} \cdot e^{j\omega t} + C_2 \cdot e^{j\beta x} \cdot e^{j\omega t} \right\}$$

$$= Re\left\{ C_1 \cdot e^{j(\omega t - \beta x)} + C_2 \cdot e^{j(\omega t + \beta x)} \right\}$$

$$\tilde{E}_y(x,t) = C_1 \cos(\omega t - \beta x) + C_2 \cdot \cos(\omega t + \beta x)$$

If $C_1 = C_2$ the two waves combine to form a simple standing wave, which does not progress.

wave length (λ) :- It is defined as the distance over which the sinusoidal waveform passes through a full cycle of 2π radians

$\therefore$ phase shift per unit length $\beta = \frac{2\pi}{\lambda}$

wave velocity :- The wave traveling in $+x$ direction is

$$\omega t - \beta x = \text{const}$$

$$\omega - \beta \cdot \frac{dx}{dt} = 0$$

$$\beta \cdot \frac{dx}{dt} = \omega \qquad \boxed{v = \frac{\omega}{\beta}}$$

where v = phase velocity

β = radians per unit length.

relation between v and λ :-

we know $\beta = \frac{2\pi}{\lambda}$

$$\therefore \beta\lambda = 2\pi$$

$$\frac{\omega}{v} \cdot \lambda = 2\pi \qquad \left(\because v = \frac{\omega}{\beta}\right)$$

$$\frac{\cancel{2\pi} f \cdot \lambda}{v} = \cancel{2\pi}$$

$$\therefore \boxed{v = f\lambda}$$

where f is measured in cycles/second or Hertz

The phase velocity $v = \frac{\omega}{\beta} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{\mu\epsilon}} = v_0$ = velocity of light.

Wave propagation in a conducting Medium :- $(\sigma \neq 0)$

The wave equation in a conducting medium is given by

$$\nabla^2 E - \gamma^2 E = 0$$

where $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$

where γ = propagation const = $\alpha + j\beta$ → phase shift const

α ↓ attenuation const

For the upw $\frac{\partial^2 E}{\partial x^2} = \gamma^2 E$

The possible solution is $E(x) = E_0 \cdot e^{-\gamma x}$

In time varying form $\tilde{E}(x,t) = \text{Re}\left\{ E_0 \cdot e^{-\gamma x + j\omega t} \right\}$

$$= e^{-\alpha x} \, \text{Re}\left\{ E_0 e^{j(\omega t - \beta x)} \right\}$$

This is the wave travelling in x direction and attenuated by $e^{-\alpha x}$

For the lossless case $\beta = \frac{2\pi}{\lambda}$ and $v = \frac{\omega}{\beta}$

$\therefore$ α = real part of $\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$ Nep/m

$$= \omega \sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right)}$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right)} \text{ rad/m}$$

Conductors and Dielectrics

The Maxwell's first equation is

$$\nabla \times H = \sigma E + j\omega\epsilon E$$

first term on right hand side is conduction current density

Second " is Displacement "

$\frac{\sigma}{\omega\epsilon} = 1$ can be considered to mark the line between conductors and Dielectrics

If $\frac{\sigma}{\omega\epsilon} \gg 1$ it is a good conductor

$\frac{\sigma}{\omega\epsilon} \ll 1$ " Dielectrics or Insulators.

$\frac{\sigma}{\omega\epsilon}$ is called as the dissipation factor D of the Dielectric.

<u>wave propagation in Good Dielectrics</u> :- For this case $\frac{\sigma}{\omega\epsilon} << 1$
(lossy dielectric)

By using Binomial expansion $\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} \cong \left(1+\frac{1}{2}\frac{\sigma^2}{\omega^2\epsilon^2}\right)$

$$\therefore \quad \alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(1+\frac{\sigma^2}{2\omega^2\epsilon^2}-1\right)} = \omega\sqrt{\frac{\mu\epsilon}{2}\cdot\frac{\sigma^2}{2\omega^2\epsilon^2}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

$$\therefore \quad \beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(1+\frac{\sigma^2}{2\omega^2\epsilon^2}+1\right)} = \omega\sqrt{\mu\epsilon}\left(1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right)$$

where $\omega\sqrt{\mu\epsilon}$ is the phase shift factor for a perfect dielectric

The velocity of the wave in

dielectric is $v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}\left(1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right)}$

$$= v_0\left(1-\frac{\sigma^2}{8\omega^2\epsilon^2}\right)$$

where v_0 = Velocity of Em wave in free space or dielectric

when $\sigma = 0$

The intrinsic or characteristic Impedance of the medium

which has finite conductivity is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}\left(\frac{1}{1+\frac{\sigma}{j\omega\epsilon}}\right)}$$

$$\simeq \sqrt{\frac{\mu}{\epsilon}}\left(1+j\frac{\sigma}{2\omega\epsilon}\right)$$

where $\sqrt{\frac{\mu}{\epsilon}} = \eta$ of dielectric when $\sigma = 0$

— o —

<u>Wave propagation in a Good conductor</u> :- $\left(\frac{\sigma}{\omega\epsilon} >> 1\right)$

The propagation constant γ is

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

$$= \sqrt{j\omega\mu \; j\omega\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)}$$

$$= j\omega\sqrt{\mu\epsilon}\left(\sqrt{1-j\frac{\sigma}{\omega\epsilon}}\right)$$

$$= j\omega\sqrt{\mu\epsilon}\sqrt{-j\frac{\sigma}{\omega\epsilon}} \qquad -j = 1\angle -90^\circ$$

$$= \frac{j\omega\sqrt{\mu\epsilon}\cdot\sqrt{\sigma}}{\sqrt{\omega\epsilon}}$$

$$\sqrt{1\angle -90^\circ} = 1\angle -45^\circ = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$= j\sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$$

$$\gamma = (j1+1)\sqrt{\pi f\mu\sigma}$$

$$\boxed{\alpha = \beta = \sqrt{\pi f\mu\sigma}}$$

Velocity of the wave in a conductor is $v = \frac{\omega}{\beta} = \frac{\omega \times \sqrt{2}}{\sqrt{\omega\mu\sigma}}$

$$= \sqrt{\frac{2\omega}{\mu\sigma}} \ m/s$$

The Intrinsic impedance of the conductor is

$$\eta \simeq \sqrt{\frac{j\omega\mu}{\cancel{\omega\epsilon}\,\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \ \Omega$$

For good conductors σ is very large and both α and β are also large. The wave is attenuated greately as it progresses through the conductor and β is also high. The characteristic Impe is also very small and has reactive component. The angle of the Impedance is 45° for a good conductor.

The Skin depth δ is Mathematically equal to $\frac{1}{\alpha}$.

$$\therefore \delta = \frac{1}{\alpha}$$

$$\delta = \frac{1}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \frac{1}{\sqrt{\frac{\cancel{2}\pi f\mu\sigma}{\cancel{2}}}} = \frac{1}{\sqrt{\pi f\mu\sigma}} \text{ metres.}$$

Skindepth $\delta \propto \frac{1}{\sqrt{f}}$ $\boxed{\delta \propto \frac{1}{\sqrt{\sigma}}}$

Hence we can write that $\boxed{\frac{\delta_1}{\delta_2} = \sqrt{\frac{f_2}{f_1}}}$

Propagation in Good conductors and Skin effect (Depth of penetrat

A medium which has conductivity the wave is attenuated as it progresses owing to losses which occur. In a good conductor at radio frequencies the rate of attenuation is very great and penetrate only a small distance before reduced to a neglegibly small % of its original strength.

Skin depth or Depth of penetration (δ) :- It is defined as that depth or distance after which $|E|$ has decreased to $\frac{1}{e}$ or 37 % of its original value. Since the Amplitude decreases by a factor $e^{-\alpha x}$.

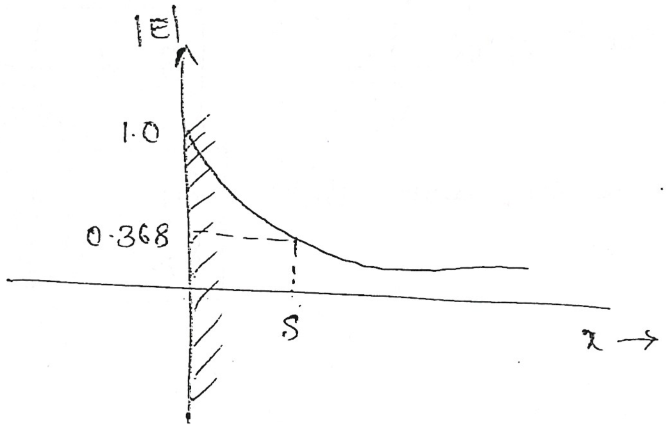
δ is defined as distance where the amplitude is $\frac{1}{e}$ times its value at $x = 0$ $\therefore$ $\alpha x = 1$

$$\therefore \delta \alpha = 1$$

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)}}$$

For a good conductor the depth of pentration is

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$



Wave propagation in different Media :-

Loss tangent $\tan\theta = \left|\frac{\sigma}{\omega\epsilon}\right|$

S.NO.	parameter	Freespace $(\epsilon=\epsilon_0, \mu=\mu_0, \sigma=0)$	perfect dielectrics $(\epsilon_0\epsilon_r, \mu_0\mu_r, \sigma=0)$	Good conductors $\left(\frac{\sigma}{\omega\epsilon} >> 1\right)$	Good Dielectrics $\left(\frac{\sigma}{\omega\epsilon} << 1\right)$
1.	Attenuation const (α) (Nep/m)	0	0	$\sqrt{\frac{\omega\mu\sigma}{2}}$	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$
2.	phase shift const (β) (rad/m)	$\omega\sqrt{\mu_0\epsilon_0}$	$\omega\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}$	$\sqrt{\frac{\omega\mu\sigma}{2}}$	$\omega\sqrt{\mu\epsilon}\left(1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right)$
3.	Propagation const (γ) $\left(\frac{1}{m}\right)$	$j\omega\sqrt{\mu_0\epsilon_0}$	$j\omega\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}$	$\sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}}$	$\alpha + j\beta$ (add above)
4.	Velocity $v = \frac{\omega}{\beta}$ m/s	$\frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8$ m/s	$\frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}}$	$\sqrt{\frac{2\omega}{\mu\sigma}}$	$v_0\left(1-\frac{\sigma^2}{8\omega^2\epsilon^2}\right)$
5.	Intrinsic Impedance (Ω) $\eta = \sqrt{\frac{\mu_0}{\epsilon_0}}\ \Omega$	120π or 377Ω	$\sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}}$	$\sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ$	$\sqrt{\frac{\mu}{\epsilon}}\left(1+j\frac{\sigma}{2\omega\epsilon}\right)$
6.	Skin depth (δ) $\delta = \frac{1}{\alpha}$ metres	∞	∞	$\frac{1}{\sqrt{\pi f\mu\sigma}}$	$\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$

Conduction and Displacement current densities :- we know that the most popular Maxwell's equation is

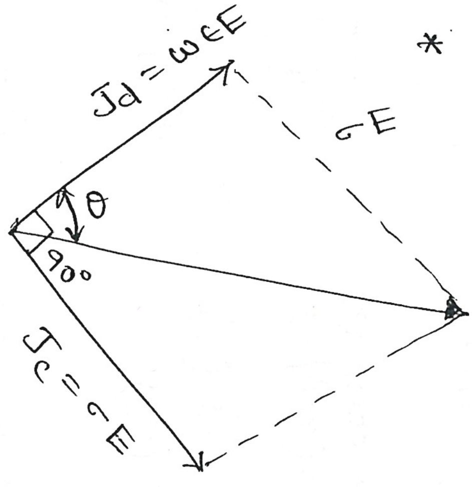
$$\nabla \times H = J + \dot{D}$$
$$= \sigma E + \epsilon \dot{E}$$
$$\nabla \times H = \underbrace{\sigma E}_{J_c} + \underbrace{j\omega \epsilon E}_{J_d} \quad \text{(using phasor notation)}$$

First term is conduction current density $J_c = \sigma \bar{E}$ and second term is Displacement current density $J_d = j\omega\epsilon\bar{E}$

Loss Tangent :- The ratio of conduction current density to displacement current density is called as Loss tangent ($\tan\theta$).

$$\tan\theta = \left|\frac{J_c}{J_d}\right| = \left|\frac{\sigma \cancel{E}}{\omega\epsilon\cancel{E}}\right| = \left|\frac{\sigma}{\omega\epsilon}\right|$$

The two current densties are displaced by an angle of 90° and Displacement current leads the conduction current. The vector diagram is given below.



* Loss tangent decides how Lossy the dielectric is

The Frequency at which both conduction and displacement current densities are equal is given by

$$J_c = J_d$$
$$\sigma \cancel{E} = \omega\epsilon\cancel{E}$$
$$\sigma = 2\pi f \epsilon$$

$$\therefore \boxed{f = \frac{\sigma}{2\pi\epsilon}}$$

Polarisation :- The polarisation of a upw refers to the time varying behav of Electric field strength vector at some fixed point in space.

a) **Linear polarisation :-**

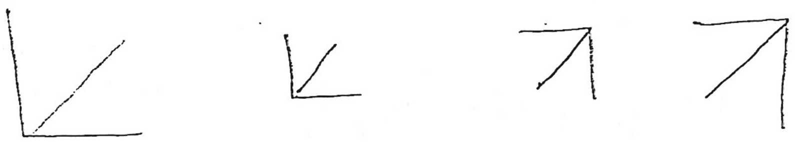
If a upw is travelling in z direction and E and H are lying in xy plane.

If $E_y = 0$ and only E_x is present the wave is said to be polarised in x direction.

If $E_x = 0$ and only E_y is present, the wave is said to be polarised in y direction.

If both E_x, E_y are present and are in phase the E field has a direction dependent on the relative magnitudes of E_x and

The angle with the x axis is $\phi = \tan^{-1}\left(\frac{E_y}{E_x}\right)$



The direction of resultant vector is const with time the wave is said to be linearly polarised.

b) **Elliptical polarisation :-** If E_x and E_y are not in phase and they reach their Max values at different instants of time, then the direction of resultant Electric vector vary with time. The end point of resultant E will be an ellipse and the wave is said to be elliptically polarised.

If the x and y components of E differs in amplitude, if y component leads x component by 90°

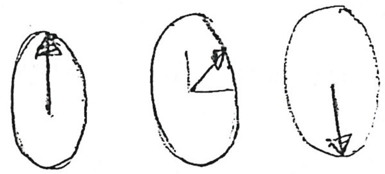
$$E_0 = A a_x + JB a_y$$

$$\tilde{E}_0(0,t) = A \cos\omega t \, a_x - B \sin\omega t \, a_y$$

The componenents of time varying fields are

$$\tilde{E}_x = A \cos\omega t \qquad \tilde{E}_y = -B \sin\omega t$$

$$\therefore \boxed{\frac{\tilde{E}_x^2}{A^2} + \frac{\tilde{E}_y^2}{B^2} = 1}$$



The end point of $\tilde{E}(0,t)$ traces out an ellipse and the wave is elliptically polarised. The sense of polarisation is left handed.

c) <u>circular polarisation</u> :- If both x and y components are equal in magnitude and if y component leads x component by 90° and

$$E_x = E_y = E_a$$

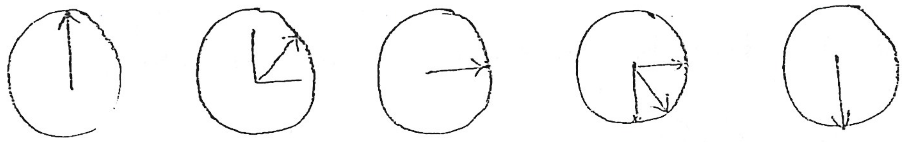
$$E_0 = (a_x + J a_y) E_a$$

$$\tilde{E}(0,t) = (a_x \cos\omega t - a_y \sin\omega t) E_a$$

$$\therefore \tilde{E}_x = E_a \cos\omega t \qquad \tilde{E}_y = -E_a \sin\omega t$$

$$\boxed{\tilde{E}_x^2 + \tilde{E}_y^2 = E_a^2}$$

The end point of $\tilde{E}(0,t)$ traces out a circle of radius.

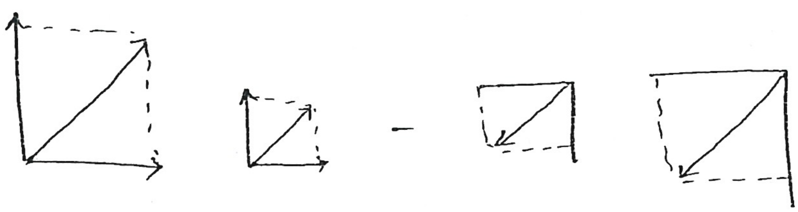
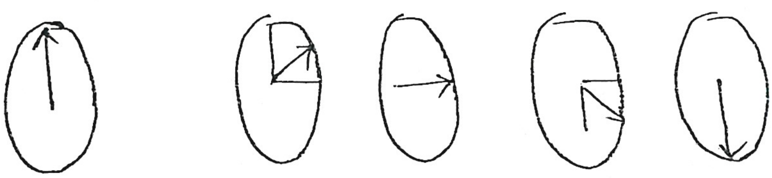


It is seen that the sense or direction of rotation is that of left hand screw advancing z direction. The wave is said to be left circularly polarised.

$$\text{If} \quad E_0 = (a_x - j a_y) E_a \quad (\text{RCP})$$

The reversal of sense of rotation may be obtained by 180° phase shift applied either to x component or y component.

We can summarise the polarisations at different times.

Type of polarisation	$t=0$	$T/8$	$\frac{T}{4}$	$\frac{3T}{8}$	$T/2$
Linear polarisation					
Elliptical polarisation					
circular polarisation